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Ordered Sets with the Standardizing Property and Straightening Laws for Algebras of Invariants

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In *Math. Z.* (**176** (1981), 359–374) I explicitly determined the invariants of a certain class of unipotent group actions, and obtained a positive partial answer to Hilbert's 14th problem for nonreductive groups. The class of groups for which the method worked remained quite obscure. Theorem (4.2) of the present paper gives a precise description of the cases where the algebras of invariants are spanned by standard bitableaux, hence have a straightening law. The unipotent groups in question ("radizielle Untergruppen" of $\mathbf{GL}_n$) correspond, up to conjugation, to finite (partially) ordered sets. The promised description is done by properties of the ordered sets that are easy to test. This is another example where combinatorial methods are important for the theory of invariants. © 1987 Academic Press, Inc.

1. INTRODUCTION: INVARIANTS OF NONREDUCTIVE GROUPS

In this introduction let k be an algebraically closed field for the sake of simplicity.

The main problem of the qualitative theory of invariants is *Hilbert's* 14th *problem*:

Let a subgroup G of $\mathbf{GL}_m$ act on the polynomial algebra $k[X] = k[X_1,..., X_m]$ in the natural way. Is the algebra $k[X]^G$ of invariants finitely generated?

From the historian's point of view it is not quite correct to denote this problem by "Hilbert's 14th problem." When Hilbert formulated his problems in 1900, he believed that Maurer had solved this problem. Accordingly he posed a more general problem that corresponded to his, then fashionable, trend of banishing the theory of invariants. I do not know when Maurer's error was detected; at all events, nowadays, the problem as stated above is in the center of interest.

For reductive (algebraic) groups the answer is positive by the results of Hilbert, Weyl, and Mumford (if k has characteristic 0), and of Nagata and Haboush (if k has arbitrary characteristic). A detailed survey is given in [10].

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For nonreductive groups only a few results are known:

(1) Nagata's counterexamples, see [10].

(2) Popov's theorem [12]: Let G be an algebraic group such that $k[Y]^G$ is finitely generated whenever G acts rationally on an affine variety Y with affine algebra $k[Y]$. Then G is reductive.

(3) Almost all known positive results can be deduced from Grosshans's theorem on invariants [7]: Let $\mathbf{GL}_n$ act on the vector space $\mathbf{M}_{nn}$ of $n \times n$ matrices, and, accordingly, on the polynomial algebra $k[\mathbf{M}_{nn}] = k[X_{ij} \mid 1 \leqslant i, j \leqslant n]$, by left translation. Let H be a closed subgroup of $\mathbf{SL}_n$ such that $k[\mathbf{M}_{nn}]^H$ is finitely generated. Let A be an arbitrary affine algebra on which $\mathbf{GL}_n$ acts rationally. Then A^H is finitely generated.

Note that an action of $\mathbf{GL}_n$ is needed. Of course, this is a disadvantage of the theorem, but should we expect more in view of the negative results above? Therefore the following seems to be a good substitute of Hilbert's 14th problem: *Find the Grosshans subgroups of* $\mathbf{GL}_n$, that is, the groups H for which $k[\mathbf{M}_{nn}]^H$ is finitely generated.

More generally, a subgroup H of the reductive group G is called a *Grosshans subgroup* [11], if H is observable and the algebra $k[G/H]$ of polynomial functions on the homogeneous space G/H is finitely generated; the technical condition "observable" should be added to the definition in the case of $\mathbf{GL}_n$ above; however, unipotent groups automatically satisfy it. There are two natural conjectures:

Conjecture 1. Each unipotent subgroup that is normalized by a maximal torus of G ("regular subgroup," "radizielle Untergruppe" in [11]) is a Grosshans subgroup.

This conjecture was formulated by Popov independently. Hochschild and Mostow [9] proved, in characteristic 0, that the unipotent radicals of the parabolic subgroups are Grosshans subgroups.

Conjecture 2. Each one-dimensional unipotent subgroup is a Grosshans subgroup.

If k has characteristic 0, this is true by Weitzenböck's theorem, compare [7, p. 250]. For a discussion of this theorem in positive characteristic see [6].

Besides the reductive subgroups only a few examples of Grosshans subgroups are known, see the list in [11, (1.6)]. The goal of [11] and the present paper is a systematical approach to Conjecture 1 for the case of $G = \mathbf{GL}_n$. Unfortunately I can give partial results only. Therefore the conjecture remains open even if $G = \mathbf{GL}_n$.

2. Remarks on Straightening and Invariants

Let k be a commutative ring with $1 \neq 0$. From (2.4) k will be infinite and entire. Let

$$A = A_N = k[X_{ij} \mid 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant N]$$

be the polynomial algebra in nN variables, sometimes called the *letter place algebra* with n letters and N places [3, 1].

(2.1) The *straightening law* of Doubilet, Rota, and Stein [4] says that *the standard bitableaux form a basis of A over k.* (There is no need to distinguish between bitableaux and bideterminants in the context of the present paper.) More precisely:

Let T be a bitableau in A. Then T is a linear combination of standard bitableaux with the properties:

(a) *They have the same content as T,*

(b) *they are dominated by the standardized bitableau T^s.*

(c) *they have at most as many rows as T.*

For (a) see [3, 2, 1].

In (b) the standardized bitableau T^s is obtained from T by ordering the columns of T increasingly (the rows of T being assumed strongly increasingly ordered; this is more practical but, of course, less aesthetical than the usage in [2], where the left half-rows are written decreasingly). The dominance relation of standard bitableaux, written $\trianglelefteq$, is the column dominance relation of [1], and is the reverse relation of that used in [2]. For the proof of (b) see [2] or [1]. The standard bitableau T^s itself has the coefficient 1.

Finally (c) is implied by (b): Look at the first column.

(2.2) In straightening a bitableau T one does not necessarily need all the standard bitableaux $\trianglelefteq T^s$. The proposition below gives a profitable reduction of the number of standard bitableaux to be considered.

Let me call rth *partial tableau*, and denote by $T^{(r)}$, the tableau consisting of the entries $\leqslant r$ of the left-hand tableau of T, where $1 \leqslant r \leqslant n$. In particular $T^{(n)}$ is the complete left-hand tableau.

Proposition. *Let T be a bitableau with two rows, and let the* rth *partial tableau $T^{(r)}$ be standard. Then T is a linear combination of standard bitableaux whose* rth *partial tableaux contain the complete first row of $T^{(r)}$ in their first rows.*

Proof. Let us straighten T. Any standard bitableau Z that possibly occurs with nonzero coefficient and violates the assertion looks as follows: The rth partial tableau $Z^{(r)}$ of Z contains an entry i in its second row that appears only in the first row of $T^{(r)}$. Let i_Z be the smallest such entry. Since the shape of $Z^{(r)}$ is at least as long as the shape of $T^{(r)}$, the emigration of i must be compensated by an entry j in the first row of $Z^{(r)}$ that appears only in the second row of $T^{(r)}$; let j_Z be the smallest such entry. Since $Z^{(r)} \trianglelefteq T^{(r)}$, we have $j_Z < i_Z$.

Now choose Z such that $i = i_Z$ is minimum and, i_Z being fixed, $j = j_Z$ is maximum. Then $T^{(r)}$ and $Z^{(r)}$ look as follows:

$$T^{(r)} = \begin{pmatrix} i_1 \cdots i_p i_{p+1} \cdots i_s i_{s+1} \cdots i_t i * \cdots \\ j_1 \cdots j_p j * \cdots \end{pmatrix},$$

$$Z^{(r)} = \begin{pmatrix} i_1 \cdots i_p i_{p+1} \cdots i_s j * \cdots \\ j_1 \cdots j_p * \cdots \qquad \cdots * i * \cdots \end{pmatrix},$$

$s \geqslant p+1$ (since $j \geqslant i_{p+1}$), and all rows strongly monotonically increasing. The entries $*$ will not be interesting. The position of i in the second row of $Z^{(r)}$ might be under j or even more on the left.

I use that part C of the Capelli operator of Z that converts the entries $\leqslant j$ of the left-hand tableaux. It transforms T and Z into

$$CT = \left(\begin{matrix} 1 \cdots p\, p+1 \cdots s\, i_{s+1} \cdots i_t i * \cdots \\ 1 \cdots p\, s+1 * \cdots \end{matrix} \,\middle|\, \begin{matrix} \cdots \\ \cdots \end{matrix}\right),$$

$$CZ = \left(\begin{matrix} 1 \cdots p p+1 \cdots s s+1 * \cdots \\ 1 \cdots p * \cdots \qquad \cdots * i * \cdots \end{matrix} \,\middle|\, \begin{matrix} \cdots \\ \cdots \end{matrix}\right).$$

What about the other occurring standard bitableaux X? Since $X \trianglelefteq T^s$, the jth partial tableau $X^{(j)}$ is dominated by

$$T^{(j)} = \begin{pmatrix} i_1 \cdots i_p i_{p+1} \cdots i_s \\ j_1 \cdots j_p j \end{pmatrix}.$$

On the other hand, if $X^{(j)}$ does not dominate

$$Z^{(j)} = \begin{pmatrix} i_1 \cdots i_p i_{p+1} \cdots i_s j \\ j_1 \cdots j_p \end{pmatrix},$$

we have $CX = 0$. Accordingly I distinguish the following five types of standard bitableaux that possibly occur in the basis expansion of T:

(a) The jth partial tableau $X^{(j)}$ does not dominate $Z^{(j)}$.

For all other types, $X^{(j)}$ is identical with one of $T^{(j)}$ or $Z^{(j)}$. As a further characteristic I consider which row contains i, and obtain the types:

$$\text{(b)}\quad X=\left(\begin{matrix} i_1\cdots i_p i_{p+1}\cdots i_s*\cdots *i*\cdots \\ j_1\cdots j_p j*\cdots \end{matrix}\;\middle|\;\begin{matrix}\cdots\\ \cdots\end{matrix}\right),$$

$$\text{(c)}\quad X=\left(\begin{matrix} i_1\cdots i_p i_{p+1}\cdots i_s*\cdots \\ j_1\cdots j_p j*\cdots \quad *i*\cdots \end{matrix}\;\middle|\;\begin{matrix}\cdots\\ \cdots\end{matrix}\right),$$

$$\text{(d)}\quad X=\left(\begin{matrix} i_1\cdots i_p i_{p+1}\cdots i_s j*\cdots *i*\cdots \\ j_1\cdots j_p *\cdots \end{matrix}\;\middle|\;\begin{matrix}\cdots\\ \cdots\end{matrix}\right),$$

$$\text{(e)}\quad X=\left(\begin{matrix} i_1\cdots i_p i_{p+1}\cdots i_s j x_{s+2}\cdots x_l \\ j_1\cdots j_p y_{p+1}\cdots y_q i y_{q+2}\cdots y_m \end{matrix}\;\middle|\;Y\right).$$

But, by the choice of Z, the type (c) does not actually occur, and Z itself is of type (e).

Now I apply the polarisation operator $D=D_{is}$ that converts i into s. It causes $DCT=0$, and $DCX=0$ for the standard bitableaux X of types (a), (b), and (d). Any standard bitableau X of type (e) is converted into a standard bitableau, up to sign:

$$DCX=(-1)^{q-p}\left(\begin{matrix} 1\cdots p\,p+1\cdots s\,s+1\,x_{s+2}\cdots x_l \\ 1\cdots p\,s\,y_{p+1}\cdots y_q y_{q+2}\cdots y_m \end{matrix}\;\middle|\;Y\right);$$

remember $x_{s+2}, y_{p+1}>j>i_s\geqslant s$. In particular X can be reconstructed from DCX. Consequently the standard bitableaux of type (e) are converted into distinct standard bitableaux. Hence the basis expansion of T,

$$T=\sum c_X X,$$

is converted into the linear combination

$$0=\sum_{\text{type(e)}} c_X\, DCX$$

of distinct standard bitableaux. I conclude $c_X=0$ if X is of type (e). In particular $c_Z=0$.

The proposition now follows by induction. ▮

(2.3) If the left-hand side of T is standard, then the occurring bitableaux of the same shape as T have also the same left-hand tableau as T. This partial result was stated without explicit proof in [11]; it follows by the shuffle product rules [3, pp. 68–70].

The following statement seems to be the natural generalization of Proposition (2.2):

Conjecture 3. Let T be a bitableau, and let the rth partial tableau $T^{(r)}$ of T be standard. Then T is a linear combination of standard bitableaux whose rth partial tableaux contain in their l top rows all entries of the l top rows of $T^{(r)}$, counted with multiplicities, for all l. ("Left-hand entries $\leqslant r$ cannot move downwards.")

Of course this is equivalent with the analogical statement about standard right-hand partial tableaux.

EXAMPLE. Consider the bitableau

$$T = \begin{pmatrix} 1247 & | & 1356 \\ 1356 & | & 1247 \end{pmatrix}.$$

There are 132 standard bitableaux $\trianglelefteq T$, 5^2 of shape (4, 4), 9^2 of shape (5, 3), 5^2 of shape (6, 2), and 1 of shape (7, 1). But we may disregard the standard bitableaux whose 5th partial tableaux are

$$\begin{pmatrix} 123 \\ 145 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1235 \\ 14 \end{pmatrix}.$$

Because T does not alter by exchange of its rows the same remark applies to the right sides. Therefore the number of bitableaux to be considered is reduced to $3^2 + 6^2 + 4^2 + 1 = 62$.

(2.4) Now I consider invariants. Accordingly I require k to be infinite entire. Of course I could let be k arbitrary and instead consider universal invariants, that is, polynomials remaining invariant when the base ring extends.

The group $\mathbf{GL}_n(k)$ of k-valued points of the group scheme $\mathbf{GL}_n$ is simply denoted by $\mathbf{GL}_n$. A *canonical unipotent subgroup* of $\mathbf{GL}_n$ [11, p. 363] is given by a subset

$$\Psi \subseteq \{(i, j) \mid 1 \leqslant i < j \leqslant n\}$$

with the property

$$(i, j), (j, l) \in \Psi \Rightarrow (i, l) \in \Psi.$$

Therefore Ψ is an ordering of $\{1, ..., n\}$, coarser than the natural ordering. The corresponding canonical unipotent subgroup $U = U_\Psi$ consists of the matrices $u = (u_{ij})$ where

$$u_{ij} = \begin{cases} 1 & \text{if } i = j, \\ \text{arbitrary} \in k & \text{if } (i, j) \in \Psi, \\ 0 & \text{else.} \end{cases}$$

The set Ψ is called the *root system* of U. Up to conjugationby a permutation matrix, U is completely determined by the isomorphism class of the ordered set $(\{1,..., n\}, \Psi)$. The group U can be described by an $n \times n$ box with entries 1 on the diagonal, $*$ at the positions $(i, j) \in \Psi$, 0 else [11], or by its Malyšev graph with edges corresponding to $1,..., n$, and with an arrow from i to j if $(i, j) \in \Psi$ and $(i, l) \notin \Psi$ for $i < l < j$.

The element $u \in U$, $u^{-1} = (u_{ij})$, acts on A by the formula

$$u \cdot X_{ij} = X_{ij} + \sum_{(i, l) \in \Psi} u_{il} X_{lj}$$

(misprinted in [11, p. 364]).

(**2.5**) A minor is a bitableau with just one row,

$$(i_1 \cdots i_m \mid j_1 \cdots j_m), \qquad 1 \leqslant i_1 < \cdots < i_m \leqslant n,$$
$$1 \leqslant j_1 < \cdots < j_m \leqslant N.$$

It is an invariant of U if and only if it has the property

$$(I_\Psi) \text{ If } (i_r, l) \in \Psi, \text{ then } l \in \{i_{r+1},..., i_m\}.$$

In this case it is called a *Ψ-minor*. A *Ψ-bitableau* is a bitableau whose rows are Ψ-minors.

Conjecture 4. The algebra A^U of invariants is generated by the (finitely many) Ψ-minors (for all N).

The significance of this conjecture for Hilbert's 14th problem would be that then all regular subgroups of $\mathbf{GL}_n$ were Grosshans subgroups, up to transition to their observable hulls, compare the introduction.

The algebra A^U contains all Ψ-bitableaux. The goal of this paper is a necessary and sufficient criterion, Theorem (4.2), that A^U is spanned by the standard Ψ-bitableaux; then A^U is an algebra with straightening law in the sense of [5]. The example in [11, p. 370], where $n = 4$ and $\Psi = \{(1, 4)\}$, shows that this is not always true. But that example is not quite serious because, by a permutation matrix, the root system is changed to $\Psi = \{(1, 2)\}$, and then the criterion (4.2) applies; the statement of the conjecture is not touched by a permutation. For serious examples see (4.3).

One might ask whether A^U could be finitely generated without being generated by the Ψ-minors; for a possible example see (4.3), Example 7. I can show that A^U, if finitely generated, must be an integral extension of the algebra R generated by the Ψ-minors. Since I have no use for this, I omit the proof. Note that A^U and R have the same quotient field [11, (2.6)].

Likewise one can consider more general regular subgroups G of $\mathbf{GL}_n$. The questions

(i) Is A^G finitely generated?

(ii) Is A^G generated by the invariant minors?

(iii) Is A^G spanned by the invariant standard bitableaux?

are easily reduced to the corresponding questions for the unipotent radicals $R_u(G)$. Therefore it is no loss to confine to the unipotent case.

3. Ordered Sets with the Standardizing or Dominance Property

In this section I consider the properties of ordered sets that are relevant for canonical unipotent subgroups and their invariants.

(3.1) Let Ω be a finite ordered set. We may describe it by an oriented graph:

$$(\sigma, \tau) \in \Omega^2 \text{ is an arrow: } \Leftrightarrow$$
$$\sigma > \tau, \text{ and } \sigma, \tau \text{ are (immediate) neighbors.}$$

The resulting graph contains no cycle all of whose arrows,except possibly one, are directed in one direction ("quasi-cycle"). Vice versa the graph determines the ordering:

$$\sigma \geqslant \tau \quad \Leftrightarrow \quad \text{There is a way (maybe trivial) from } \sigma \text{ to } \tau.$$

This correspondence between finite ordered sets (up to isomorphism) and finite directed graphs without quasi-cycles is bijective.

If $\sigma \in \Omega$, the set

$$\Omega_\sigma := \{\tau \in \Omega \mid \tau < \sigma\}$$

is called the *segment below* σ.

An *admissible enumeration* of Ω is an antitone bijection

$$\Omega \to \{1,..., n\}, \qquad \sigma_i \leftrightarrow i,$$

or, in other words, a total ordering refining the reverse of the given ordering. (σ_1 is the greatest element of Ω.)

(3.2) Let Ω be an ordered set of n elements with an admissible enumeration. Let Ψ be the induced reverse strict ordering of $\{1,..., n\}$,

$$\Psi = \{(i, j) \in \{1,..., n\}^2 \mid \sigma_i > \sigma_j\}.$$

A Ψ-*tableau* is a tableau with values in $\{1,..., n\}$ and strictly increasing rows such that the following holds: If a row contains the entry i, then it contains all j with $(i, j) \in \Psi$. (Accordingly the Ψ-tableaux are the left sides of the Ψ-bitableaux as defined in (2.5).)

For a finite ordered set Ω with an admissible enumeration the following properties are relevant:

(S) If T is a Ψ-tableau, then the standardized tableau T^s is again a Ψ-tableau.

(D) If T is a Ψ-tableau and S is a standard tableau dominated by, and of the same content as, T, then S is a Ψ-tableau.

Here again T^s results from T by ordering the columns. The dominance relation of tableaux, written $\trianglelefteq$, is the column dominance relation, compare (2.1). Since T has increasing rows, $S \trianglelefteq T$ is equivalent with $S \trianglelefteq T^s$.

DEFINITION. A finite ordered set has the *standardizing property* (or the *dominance property*), if it has an admissible enumeration with the property (S) (or (D)).

Remarks. (1) It suffices to test (S) for Ψ-tableaux with two rows: The rearrangement of T yielding T^s decomposes into single steps involving only two rows. Since each standard tableau $S \trianglelefteq T$ of the same content is also got by rearrangement, a similar remark applies to (D).

(2) In [11] I considered the following property:

(A) If T is a Ψ-bitableau, then T is a linear combination of standard Ψ-bitableaux.

Obviously (D) $\Rightarrow$ (A) $\Rightarrow$ (S). In (4.1) I shall show that, in fact, (A) and (S) are equivalent, whereas (D) is strictly stronger.

Since, given a finite ordered set, there may be a lot of admissible enumerations, I shall give criteria that depend only on the ordering and yield almost canonical enumerations, see Theorems (3.8) and (3.9).

(**3.3**) LEMMA. *Let Ω be a finite ordered set, admissibly enumerated, and let Ω_i be the segment below σ_i. Assume* (S) *holds. Then:*

(i) $\Omega \supset \Omega_1 \supseteq \cdots \supseteq \Omega_n = \varnothing$.

(ii) *If $\sigma_p, \sigma_q \in \Omega - \Omega_i$, where $i < p < q \leqslant n$, then*

$$\Omega_i \cap \{\sigma_{q+1},..., \sigma_n\} \subseteq \Omega_{i+1}.$$

(iii) *Assume moreover that* (D) *holds, and $\sigma_p \in \Omega - \Omega_i$ where $i < p \leqslant n$. Then*

$$\Omega_i \cap \{\sigma_{p+1},..., \sigma_n\} \subseteq \Omega_{i+1}.$$

Proof. (i) Assume there is an i with $\Omega_i \not\supseteq \Omega_{i+1}$. Let $\sigma_m \in \Omega_{i+1} - \Omega_i$, in particular $i+2 \leqslant m \leqslant n$. Then necessarily $\sigma_{i+1} \notin \Omega_i$, since otherwise $\sigma_i > \sigma_{i+1} > \sigma_m$. Therefore

$$T = \begin{pmatrix} i+1 & i+2 & \cdots & m-1 & m & \cdots & n-1 & n \\ i & i+2 & \cdots & m-1 & m+1 & \cdots & n & \end{pmatrix}$$

is a Ψ-tableau. The standardized tableau,

$$T^s = \begin{pmatrix} i & i+2 & \cdots & m-1 & m & \cdots & n-1 & n \\ i+1 & i+2 & \cdots & m-1 & m+1 & \cdots & n & \end{pmatrix},$$

however, is not a Ψ-tableau, since it has $i+1$ in its second row but not m.

(ii) Let $i_1 < \cdots < i_u$ be the numbers of the elements of Ω_i. By (i) the numbers of the elements of Ω_{i+1} are contained among these. Without loss of generality I can assume p and q minimum and $q < i_u$.

If the assertion is wrong, I can find an $i_t > q$, $t \leqslant u$, with $\sigma_{i_t} \in \Omega_i - \Omega_{i+1}$. Let $i_0 = i$, and let r and s be the indices such that $i_r < p < i_{r+1}$, $i_s < q < i_{s+1}$ (where $0 \leqslant r \leqslant s < t \leqslant u$). The Ψ-tableau

$$T = \begin{pmatrix} i & i_1 & \cdots & i_{r-1} & i_r & i_{r+1} & \cdots & i_s & i_{s+1} & i_{s+2} & \cdots & i_t & i_{t+1} & \cdots & i_u \\ i+1 & i_2 & \cdots & i_r & p & i_{r+1} & \cdots & i_s & q & i_{s+1} & \cdots & i_{t-1} & i_{t+1} & \cdots & i_u \end{pmatrix}$$

(suitable interpreted when $r=0$, or $r=1$, or $r=s$, or $t=s+1$, or $t=u$) yields the standardized tableau

$$T^s = \begin{pmatrix} i & i_1 \cdots i_{r-1} & i_r & i_{r+1} \cdots i_s & q & i_{s+1} \cdots i_{t-1} & i_{t+1} \cdots i_u \\ i+1 & i_2 \cdots & i_r & p \; i_{r+1} \cdots i_s & i_{s+1} & i_{s+2} \cdots & i_t \;\; i_{t+1} \cdots i_u \end{pmatrix}$$

that is not a Ψ-tableau: It has i in its first row but not i_t.

(iii) Assume p minimum, $i_s < p < i_{s+1}$, and $\sigma_{i_t} \in \Omega_i - \Omega_{i+1}$ where $s+1 \leqslant t < u$. Then the Ψ-tableau

$$T = \begin{pmatrix} i & i_1 \cdots i_{s-1} & i_s & i_{s+1} \cdots i_{t-1} & i_t & \cdots i_{u-1} & i_u \\ i+1 & i_1 \cdots i_{s-1} & p & i_{s+1} \cdots i_{t-1} & i_{t+1} & \cdots & i_u \end{pmatrix}$$

dominates the standard tableau

$$S = \begin{pmatrix} i & i_1 \cdots i_{s-1} & i_s & p & i_{s+1} & \cdots & i_{t-1} & i_{t+1} & \cdots & i_{u-1} & i_u \\ i+1 & i_1 \cdots i_{s-1} & i_{s+1} & \cdots & & & i_t & i_{t+1} & \cdots & & i_u \end{pmatrix}$$

that is not a Ψ-tableau: It has i in its first row but not i_t. ∎

(**3.4**) *Remarks.* (1) The following statements are equivalent:

(a) There is an admissible enumeration such that the condition (i) of Lemma (3.3) holds.

(b) Any two segments are comparable.

(c) For any two arrows of the associated graph there must be a (directed) way from at least one of the initial points to the other final point. (In particular the graph must be connected up to some isolated points.)

Proof. (a) ⇒ (c) If (c) is not fulfilled, then any admissible enumeration obviously violates (a).

(c) ⇒ (b) If $\sigma_i, \sigma_j \in \Omega$, $\sigma_p \in \Omega_i - \Omega_j$, $\sigma_q \in \Omega_j - \Omega_i$, then there must be neighbors σ_r of σ_i and σ_s of σ_j such that $\sigma_r \notin \Omega_j$ and $\sigma_s \notin \Omega_i$.

(b) ⇒ (a) Obvious.

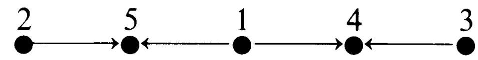
(2) Looking at the graph one can rephrase statements (ii) and (iii) of Lemma (3.3): If there is no way from σ_i to two vertices with higher number (or to one vertex with higher number), then σ_{i+1} is connected with the complete remainder of the segment below σ_i.

(3) More generally one can define the *bypass index* of a finite ordered set with admissible enumeration to be the least number b such that one has: If $\sigma_{i_1},..., \sigma_{i_b}, \sigma_p \in \Omega - \Omega_i$ where $i < i_1 < \cdots < i_b < p \leqslant n$, then

$$\Omega_i \cap \{\sigma_{p+1},..., \sigma_n\} \subseteq \Omega_{i+1}.$$

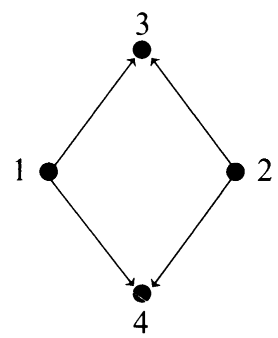
Accordingly b is the maximum number of vertices of higher rank that can be bypassed without consequences. Then (ii) means: (S) implies $b = 1$, and (iii): (D) implies $b = 0$.

(**3.5**) EXAMPLES. (1) The statements (a), (b), and (c) of Remark 1 in (3.4) do not hold, but the graph is connected:



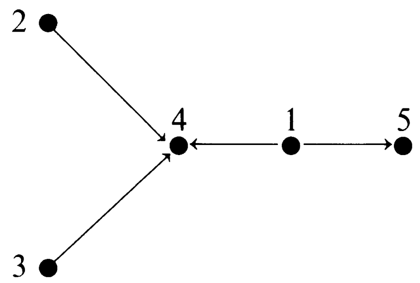
$\Omega_2 = \{\sigma_5\}$, $\Omega_3 = \{\sigma_4\}$.

(2) The statements (a), (b), and (c) hold:



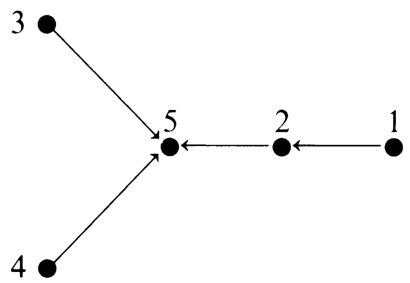
$\Omega_1 = \Omega_2 = \{\sigma_3, \sigma_4\}$, $\Omega_3 = \Omega_4 = \varnothing$.

(3) The statement (i) of Lemma (3.3) holds but not (ii):



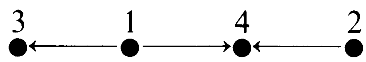
$\sigma_2, \sigma_3 \notin \Omega_1, \sigma_5 \in \Omega_1 - \Omega_2$.

(4) The statements (i), (ii), and (iii) hold:



$\Omega_1 = \{\sigma_2, \sigma_5\}$, $\Omega_2 = \Omega_3 = \Omega_4 = \{\sigma_5\}, \Omega_5 = \varnothing$. For another example see Example 2.

(5) The statements (i) and (ii) hold but (iii) does not:



$$\sigma_2 \notin \Omega_1, \Omega_1 \cap \{\sigma_3, \sigma_4\} = \{\sigma_3, \sigma_4\}, \Omega_2 = \{\sigma_4\}.$$

With the indicated enumerations, (S) does not hold for Examples 1 and 3, and (D) does not hold for Example 5. From (3.8) and (3.9) it will follow that (D) holds for Examples 2 and 4, and (S) holds for Example 5.

(3.6) The finite ordered set Ω carries a natural equivalence relation, called *segment equivalence:*

$$\sigma \sim \tau: \quad \Leftrightarrow \quad \Omega_\sigma = \Omega_\tau.$$

The segment (equivalence) classes are the fibers of the map $\Omega \to \mathfrak{P}(\Omega)$, $\sigma \mapsto \Omega_\sigma$.

If any two segments are comparable, the segment classes $M_1, \ldots, M_r$ may be enumerated such that

$$\sigma \in M_i, \qquad \tau \in M_{i+1} \Rightarrow \Omega_\tau \subset \Omega_\sigma.$$

If Ω has the standardizing property, then for any admissible enumeration with (S) all elements of M_i must have smaller numbers than all elements of M_{i+1}.

PROPOSITION. *Let Ω be a finite ordered set with the standardizing property. Then the segment class decomposition*

$$\Omega = M_1 \cup \cdots \cup M_r$$

has the properties:

(i) *Each $\sigma \in M_i$ has the same segment Ω_i below it, and*

$$\Omega_i \subseteq M_{i+1} \cup \cdots \cup M_r,$$
$$\Omega \supset \Omega_1 \supset \cdots \supset \Omega_r = \varnothing.$$

(ii) *For all i, j with $1 \leqslant i < j < r$ we have: If more than one element of $M_{i+1} \cup \cdots \cup M_j$ are absent from Ω_i, then the elements of M_i have no neighbors in $M_{j+1}, \ldots, M_r$.*

Proof. (i) Obvious by statement (i) of Lemma (3.3) and by the remark above.

(ii) Choose an admissible enumeration with (S). Let $\sigma_p, \sigma_q \in (M_{i+1} \cup \cdots \cup M_j) - \Omega_i$ where p and q are minimum. Apply the statement (ii) of Lemma (3.3) to the last element of M_i (instead of the σ_i in the lemma). It produces

$$\Omega_i \cap (M_{j+1} \cup \cdots \cup M_r) \subseteq \Omega_i \cap \{\sigma_{q+1}, \ldots, \sigma_n\} \subseteq \Omega_{i+1}.$$

If $\Omega_i \cap M_{i+1} \neq \varnothing$, then $\tau \in \Omega_i \cap M_{i+1}$ ranges between each element of M_i and each element of $\Omega_i \cap (M_{j+1} \cup \cdots \cup M_r)$.

If $\Omega_i \cap M_{i+1} = \varnothing$, we have $|M_{i+1}| = 1$; otherwise the statement (ii) of the lemma would produce the contradiction $\Omega_i = \Omega_{i+1}$. Therefore $M_{i+1} = \{\sigma_p\}$, and, without loss of generality, $\sigma_q \in M_j$ (diminish j, if necessary). Since $\Omega_i \neq \Omega_{i+1}$, necessarily $\sigma_{p+1}, \ldots, \sigma_{q-1} \notin \Omega_{i+1}$ (and $q \neq p+1$). Hence $\sigma_t \notin \Omega_{i+1}$ where $p < t < q$ (and $q \geqslant p+2$). The lemma applied to σ_p yields

$$\Omega_i \cap (M_{j+1} \cup \cdots \cup M_r) = \Omega_{i+1} \cap (M_{j+1} \cup \cdots \cup M_r) \subseteq \Omega_{i+2}.$$

Therefore $\sigma_{p+1} \in \Omega_i \cap M_{i+2}$ ranges between each element of M_i and each element of $\Omega_i \cap (M_{j+1} \cup \cdots \cup M_r)$. ∎

(**3.7**) If the segment class decomposition

$$\Omega = M_1 \cup \cdots \cup M_r$$

has the properties (i) and (ii) of Proposition (3.6), it is called *good.* A necessary condition for this is that any two segments are comparable.

EXAMPLES (numbered as in (3.5)). (1) The decomposition is not good because there are incomparable segments.

(2) Good.

(3) Not good; the segment classes are $M_1 = \{\sigma_1\}$, $M_2 = \{\sigma_2, \sigma_3\}$, $M_3 = \{\sigma_4, \sigma_5\}$.

(4) Good.

(5) Good.

Now let Ω have all its segments comparable. Then it has a *distinguished enumeration* that is carried out as follows:

(1) The elements of M_1 obtain the numbers $1, \ldots, |M_1|$, the elements of M_2 the numbers $|M_1| + 1, \ldots, |M_1| + |M_2|$, and so on.

(2) Each of the sets M_i is filtered:

$$M_i = M_{i0} \supseteq M_{i1} \supseteq \cdots \supseteq M_{i,i-1} \supseteq M_{ii} = \emptyset,$$

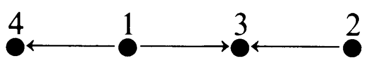
where $M_{ij} := \Omega_j \cap M_i$ for $1 \leqslant i \leqslant j-1$. In particular the segment below $\sigma \in M_j$ is

$$\Omega_\sigma = \Omega_j = M_{j+1,j} \cup \cdots \cup M_{rj}.$$

Within M_i the elements of $M_{i,i-1}$ obtain the smallest numbers (in arbitrary order), then the remaining elements of $M_{i,i-2}$ obtain the next smallest numbers, and so on.

Each distinguished enumeration is admissible.

EXAMPLES. The indicated enumerations of the Examples 2, 3, 4 are distinguished, but that of Example 5 is not. A distinguished enumeration of Example 5 is



(**3.8**) THEOREM. *Let Ω be a finite ordered set. Then the following statements are equivalent:*

(i) *Ω has the standardizing property.*

(ii) *The segment class decomposition of Ω is good.*

Proof. (i) $\Rightarrow$ (ii) is Proposition (3.6).

(ii) $\Rightarrow$ (i) I supply Ω with a distinguished enumeration and claim that (S) is true.

For the proof let T be a Ψ-tableau with two rows. Denote the entries in the first column by p and q where $\sigma_p \in M_i$, $\sigma_q \in M_j$, $i \leqslant j$, $p \leqslant q$. (It is not specified whether p or q is in the first row.)

Case 1. $j = i$. The numbers of all elements of Ω_i appear in each row of T and in each row of T^s. Because only numbers of elements of $M_i \cup \cdots \cup M_r$ can occur altogether, each entry has its obligatory companions in Ω_i, and these are in the same row of T^s.

Case 2. $j \geqslant i+1$. All entries of T belonging to $M_i \cup \cdots \cup M_{j-1}$ must be in the first row of T^s. In particular this row begins with p. If p retains its obligatory companions, so do all entries belonging to $M_i \cup \cdots \cup M_{j-1}$. On the other hand all entries belonging to $M_j \cup \cdots \cup M_r$ have their companions in $\Omega_j \subseteq \Omega_i$, and these occur in each row of T^s. Therefore I only have to show: T^s has each number belonging to $\Omega_i \cap (M_j \cup \cdots \cup M_r)$ in its first row.

This is almost obvious for the numbers belonging to $\Omega_i \cap M_j = M_{ji}$: Assume that the elements of M_j have the numbers $s+1,..., u$ (where $s+1 \leqslant q \leqslant u$), in particular that the elements of M_{ji} have the numbers $s+1,..., t$ (where $t \leqslant u$). Then the entries belonging to M_{ji} in the p-row of T are over (or under) greater entries in the q-row. Therefore they are in the first row of T^s.

Finally assume that the number t of an element of $\Omega_i \cap M_l = M_{li}$, with $l \geqslant j+1$, is missing in the first row of T^s. This number t is in the p-row of T and in a column with number at least

$$1 + |\Omega_i \cap (M_{i+1} \cup \cdots \cup M_{l-1})| + t - s,$$

where $s = |M_1 \cup \cdots \cup M_{l-1}| \geqslant q$. In the same column, and in the q-row, there must appear an entry $u < t$, and the number of this column is at most $1 + u - q$. Comparing these two estimates of the column number I get

$$|\Omega_i \cap (M_{i+1} \cup \cdots \cup M_{k-1})| + t - u \leqslant s - q,$$

where, of course,

$$s-q \leqslant |M_j \cup \cdots \cup M_{l-1}| - 1.$$

Because $t-u \geqslant 1$, I have

$$|\Omega_{j-1} \cap (M_j \cup \cdots \cup M_{l-1})| + 1 \leqslant |\Omega_i \cap (M_{i+1} \cup \cdots \cup M_{l-1})| + t - u$$
$$\leqslant |M_j \cup \cdots \cup M_{l-1}| - 1. \qquad (*)$$

Therefore at least two elements of $M_j \cup \cdots \cup M_{l-1}$ are missing in Ω_{j-1}. Since the decomposition of Ω is good, σ_t is not a neighbor of the elements of M_{j-1}. Since t is missing in the q-row of T, I have $\sigma_t \notin \Omega_j$, and accordingly $\sigma_t \notin \Omega_{j-1}$. On the left and right sides of $(*)$ I can replace the index j by any index h with $i+1 \leqslant h \leqslant j$ because $s-q$ remains between them. Hence σ_t is not a neighbor of the elements of M_{h-1}. Therefore $\sigma_t \notin \Omega_{j-2}, \ldots, \sigma_t \notin \Omega_i$, a contradiction. ▌

(3.9) With regard to the dominance property I can state a similar theorem. If the segment class decomposition has the properties (i) and (ii) of Proposition (3.6), but with "more than one" replaced by "at least one," it is called *very good.*

THEOREM. *Let Ω be a finite ordered set. Then the following statements are equivalent:*

(i) *Ω has the dominance property.*

(ii) *The segment class decomposition of Ω is very good.*

This theorem is not needed in the sequel. The proof is similar to that of Theorem (3.8) and is left to the reader.

COROLLARY. *If Ω has the standardizing (or dominance) property, then* (S) *(or* (D)*) holds for each distinguished enumeration.*

EXAMPLES. Compare the remark at the end of (3.5).

4. STRAIGHTENING LAWS FOR ALGEBRAS OF INVARIANTS

(4.1) PROPOSITION. *Let Ω be a finite ordered set with the standardizing property, supplied with an admissible enumeration such that* (S) *holds. Then each Ψ-bitableau is a linear combination of standard Ψ-bitableaux.*

Proof. I have to straighten a Ψ-bitableau with two rows,

$$T=\begin{pmatrix} p & p_1 & \cdots & \cdots & p_l & \Big| & \cdots \\ q & q_1 & \cdots & q_m & & \Big| & \end{pmatrix}$$

(maybe $m \geqslant l$), where $p \leqslant q$, $\sigma_p \in M_i$, $\sigma_q \in M_j$, $i \leqslant j$. The segment class decomposition $\Omega = M_1 \cup \cdots \cup M_r$ is good. Let S be one of the occuring bitableaux. Then S has all its (left-hand) entries $<q$ in its first row, and its second row begins with an entry $\geqslant q$. Accordingly I have only to show that S contains each number belonging to $\Omega_i \cap (M_j \cup \cdots \cup M_r)$ in its first row, compare the proof of Theorem (3.8).

If $i=j$, this is obvious because the numbers belonging to Ω_i appear twice. Now let $j \geqslant i+1$. Let $h := |M_1 \cup \cdots \cup M_{j-1}|$. Then $p \leqslant h < q$.

Case 1. At most one element of $M_j \cup \cdots \cup M_r$ with number $\leqslant$ the maximum number of Ω_i is missing in Ω_i. Let s be the maximum index with $p_s \leqslant h+s+1$. In particular all numbers belonging to Ω_i are $\leqslant p_s$. For $1 \leqslant t \leqslant \min\{s, m\}$ I have

$$p_t \leqslant h+t+1 < q+t+1 \leqslant q_t + 1,$$

hence $p_t \leqslant q_t$. Therefore the p_sth partial tableau of T is standard. By Proposition (2.2) no number belonging to Ω_i can emigrate to the second row of S.

Case 2. At least two elements of $M_j \cup \cdots \cup M_r$ with numbers $\leqslant$ the maximum number of Ω_i are missing in Ω_i. Denote the smallest two of these numbers by u, v with $u<v$. Then σ_u, σ_v are missing in $\Omega_{i+1},..., \Omega_{j-1}$, too. Let s be the index with $p_s < v \leqslant p_{s+1}$. Then, as in Case 1, I have $p_t \leqslant q_t$ for $1 \leqslant t \leqslant \min\{s, m\}$. Again the p_sth partial tableau of T is standard, and no number $<v$ belonging to Ω_i can emigrate to the second row of S. What about numbers $\geqslant v$ belonging to Ω_i? Because of property (S) and Lemma (3.3) these belong to $\Omega_{i+1},..., \Omega_j$. Therefore they appear in each row of T, hence of S. ∎

(4.2) Now let U be a canonical unipotent subgroup of $\mathbf{GL}_n$ with root system Ψ. Let U act on the algebra $A = A_N$ as in (2.4). Let Ω be an n-element ordered set inducing the ordering Ψ of $\{1,..., n\}$ by an appropriate admissible enumeration. Call U *good*, if Ω has the standardizing property and the admissible enumeration corresponding to Ψ has the property (S). This use of the term "good" does not exactly correspond to that in [11, p. 368], where moreover the formulation of Lemma (4.2) was not consistent. All the examples of [11, (3.4) and (3.7), Bemerkung 2] are good in the new sense.

THEOREM. *Let U be a canonical unipotent subgroup of $\mathbf{GL}_n$ with root system Ψ. Then the following statements are equivalent:*

(i) *The subgroup U is good.*

(ii) *The algebra A^U of invariants is spanned by the standard Ψ-bitableaux (for every N).*

Proof. (ii) $\Rightarrow$ (i) By Remark 2 in (3.2).

(i) $\Rightarrow$ (ii) Let R be the algebra generated by the Ψ-minors. Proposition (4.1) tells me that R is spanned by the standard Ψ-bitableaux. Therefore I have to show that $R = A^U$.

Let Ω be a corresponding ordered set, supplied with an enumeration with property (S) that induces Ψ. Let $\Omega = M_1 \cup \cdots \cup M_r$ be the segment class decomposition. I reorder Ω in a coarser way by forgetting all the relations $\sigma < \tau$ where $\sigma \in M_{r-1}$. This new ordered set Ω' corresponds to a canonical unipotent subgroup U' that is normal in U and has the root system

$$\Psi' = \{(i, j) \in \Psi \mid \sigma_i \notin M_{r-1}\}.$$

By Theorem (3.8) Ω' has the standardizing property. (But note that the given enumeration need not have property (S) with respect to Ω', even if it is distinguished with respect to Ω.) By induction over r, beginning at $r = 1$, I can assume that $A^{U'}$ is generated by the Ψ'-minors: Conjugation by a suitable permutation matrix transforms U' into a good subgroup U'' with root system Ψ'', and $A^{U''}$ is spanned by the standard Ψ''-bitableaux. By [11, (2.6), Bemerkung] I got $A[1/d]^U = R[1/d]$ with $d = (i_1 \cdots i_m \mid i_1 \cdots i_m)$, where $i_1, ..., i_m$ are the numbers corresponding to the last nontrivial segment Ω_{r-1}. Here I assumed that $N \geqslant n$; because the right-hand entries of the bitableau d are unessential,the assumption $N \geqslant m$ would suffice.

To complete the proof I only have to get rid of the denominator d, that is, I have to show the property (C_d) of [11, (3.3)]: If T is a standard bitableau, and the standardization S of the bitableau Td is a Ψ-bitableau, then T is a Ψ-bitableau.

To this end let

$$T = \begin{pmatrix} T' \\ T'' \end{pmatrix}$$

be decomposed into a product of two standard bitableaux where T' consists of the rows of T that contain entries belonging to $M_1 \cup \cdots \cup M_{r-1}$ on their left sides, and T'' consists of the rows containing entries only from M_r on the left. If $\binom{T'}{d}$ is standard, then the standardization of

$$Td = \begin{pmatrix} T' \\ d \\ T'' \end{pmatrix}$$

affects the lower part $\binom{d}{T''}$ only. Therefore, in this case, S is a Ψ-bitableau if and only if T' is if and only if T is.

Now assume that $\binom{T'}{d}$ is not standard. Let $(h_1 \cdots h_l | \cdots)$ be the last row of T'. Then $h_1 \in M_1 \cup \cdots \cup M_{r-1}$, hence $h_1 < i_1$. After standardizing, h_1 must have at least the elements $i_1, ..., i_m \in \Omega_{r-1}$ in its row. But i_1 can appear there only if it was there in T already. Therefore $h_2 \leqslant i_1 < i_2$, and similarly $h_s < i_s$ for $1 \leqslant s \leqslant q := \min\{l, m\}$. Hence h_1 can't get i_q, and S is not a Ψ-bitableau. ∎

Remark. It suffices to have (ii) for only one fixed value $N \geqslant n$. Even $N \geqslant n-2$ suffices.

(**4.3**) EXAMPLES. For the Example 1 of (3.5) the algebra of invariants is not spanned by the standard Ψ-bitableaux, whatever admissible enumeration is chosen. I don't even know whether this algebra is finitely generated, compare [11, (4.3)]. The only other example with $n \leqslant 5$ where the finite generation of the invariants is unknown is obtained by reversing the four arrows of Example 1.

Examples 2, 4, and 5 of (3.5) give good groups.

Example 3 gives no good group with any enumeration. But the algebra of invariants is generated by the invariant minors for the following reason: Reversing the arrows, that is, applying the transposition automorphism of $\mathbf{GL}_n$, gives a good group when the enumeration is properly chosen.

(6) For $n \leqslant 4$ there is only one example,

with incomparable segments. However in this case the algebra of invariants is generated by the invariant minors, apply [11, (4.2)]. All the other graphs with $n \leqslant 4$ give good groups, at least with a proper enumeration.

(7) Another example with incomparable segments is

look at the first and last arrows. In this case the algebra of invariants is finitely generated by [11, (4.3)]. (The base ring extension $k \subseteq$ the algebraic closure of the quotient field of k doesn't matter.) But I do not know whether the invariant minors are a system of generators.

Acknowledgments

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Note added in proof. Further results on conjecture 1 are given by Grosshans in *Invent. Math.* **73** (1983), pp. 1–9, and *Math. Z.* **193** (1986), pp. 95–103.

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