INVARIANTS OF UNIPOTENT GROUPS
A survey*

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15 August 1985, typo corrected on 2 August 2007

I'll give a survey on the known results on finite generation of invariants for nonreductive groups, and some conjectures.

You know that Hilbert's 14th problem is solved for the invariants of reductive groups; see [12] for a survey. So the general case reduces to the case of unipotent groups. But in this case there are only a few results, some negative and some positive.

I assume that k is an infinite field, say the complex numbers, but in most instances an arbitrary ring would do it.

*Appeared in S. S. Koh (Ed.): *Invariant Theory*, Springer Lecture Notes in Mathematics 1278 (1987), 8–17 — Footnotes added after publication

1 BASIC RESULTS

1.1 Nagata's counterexample (1958)

Let U be a subgroup of the n-fold product $\mathbb{G}_a^n$ of the additive group, canonically embedded in $\mathbb{GL}_{2n}$

$$U \subseteq \begin{pmatrix} \boxed{\begin{matrix} 1 & * \\ 0 & 1 \end{matrix}} & & \\ & \ddots & \\ & & \boxed{\begin{matrix} 1 & * \\ 0 & 1 \end{matrix}} \end{pmatrix} \subseteq \mathbb{GL}_{2n},$$

such that U is given by 3 'general' linear relations. Then $k[X]^U$ is not finitely generated, where $X = (X_1, ..., X_{2n})$, if n is a square $= r^2 \geq 16$ (at least if k contains enough transcendental elements).

Cf. [14]. All known counterexamples derive from this one!

Chudnovsky claims, but apparently never published a proof, that $n \geq 10$ suffices. The argument in [1] is not convincing, but there is more evidence in [2] and [15]. For the proof (with $n \geq 10$) one needs the following result:

There is a set S of n points in the affine plane $\mathbb{A}^2$ with the property: Each nonzero polynomial $f \in k[Y_1, Y_2]$ that vanishes of order at least t in each $p \in S$ has degree $d > t \cdot \sqrt{n}$ (t any integer ≥ 1).

Let $\omega_t(S)$ be the minimum of the degrees of such polynomials; then the assertion is $\omega_t(S) > t \cdot \sqrt{n}$. Now the quotient $\omega_t(S)/t$ decreases to a limit $\Omega(S)$ when t goes to infinity; $\Omega(S)$ is called the singular degree of S. In general $\Omega(S) \leq \sqrt{n}$. Chudnovsky's claim is:

If S is generic, then $\Omega(S) = \sqrt{n}$.

This gives $\omega_t(S) \geq t \cdot \sqrt{n}$. However, if n is not a square, we have the desired strict inequality because $\omega_t(S)$ is an integer. And in the case where n is a square, we can take Nagata's argument.

1.2 Popov's theorem (1979)

Popov's theorem is the converse of the invariant theorem for reductive groups. So:

For an affine algebraic group G the following statements are equivalent:

(i) G is reductive.

(ii) Whenever G acts rationally on a finitely generated algebra A, then the invariant algebra A^G is finitely generated.

See [19]. This means that for a nonreductive group there can't be a general positive answer.

1.3 Zariski's result (1954)

A positive result goes back to Zariski:

If a group G acts on a finitely generated algebra A such that the invariant algebra A^G has transcendence degree at most 2, then A^G is finitely generated.

Cf. [14]. A useful geometric version is:

COROLLARY 1 *If an affine algebraic group G acts on an affine variety X and there is an orbit of codimension ≤ 2, then $k[X]^G$ is finitely generated.*

Proof. Assume (without loss of generality) that X is normal. Then

$$\operatorname{trdeg} k[X]^G \leq \dim X - \max\{\dim G \cdot x \mid x \in X\} \leq 2.$$

COROLLARY 2 *If* $\operatorname{trdeg} A \leq 3$*, then A^G is finitely generated.*

Proof. Assume that G acts effectively. If G is finite, we are done. Else $\operatorname{trdeg} A^G \leq 2.\diamond$

For linear actions we can do one more step:

COROLLARY 3 *If G acts linearly on the polynomial algebra $k[X] = k[X_1, X_2, X_3, X_4]$, then $k[X]^G$ is finitely generated.*[1]

Proof. Assume that G acts effectively. Without changing $k[X]^G$ we may assume that G is Zariski-closed in $\mathbb{GL}_4$. If G is finite, we are done. Else $\dim G \geq 1$ and G is reductive or has a 1-dimensional unipotent normal subgroup N. The algebra $A = k[X]^N$ is finitely generated by Weitzenböck's theorem (see below 2.1b), $\operatorname{trdeg} A \leq 3$, $k[X]^G = A^G$. $\diamond$

[1] Only if $\operatorname{char} k = 0$, see footnote 2

1.4 Grosshans's principle (1973)

Some other positive results derive from **Grosshans's principle** [6]:

> *Let an algebraic group G act rationally on a k-algebra A, and H be a closed subgroup of G. Then*

$$A^H \cong (k[G]^H \otimes A)^G.$$

For the *proof* let $G \times H$ act on $k[G] \otimes A$ as follows: G acts diagonally by left translation and H acts on $k[G]$ by right translation. Then take the invariants in the two possible different ways (using an obvious isomorphism).$\diamond$

If G is reductive and A finitely generated, this reduces the question, whether A^H is finitely generated, to the one algebra $k[G]^H$ that is also the global coordinate algebra of the homogeneous space G/H.

2 APPLICATIONS OF THE GROSSHANS PRINCIPLE

For ring theoretic properties of A^H it may be useful to look at the isomorphism of 1.4. For example an unpublished result of Boutot is:

> *Let* $\operatorname{char} k = 0$ *and G reductive, acting on a finitely generated k-algebra B with only rational singularities. Then B^G also only has rational singularities; in particular B^G is Cohen-Macaulay.*

The question whether $k[G]^H$ has rational singularities, seems to be rather difficult, and I don't dare making a conjecture; but there are some known examples. If that holds, and A only has rational singularities, then also $k[G]^H \otimes A$ and hence A^H only have rational singularities.

The Grosshans principle has several important special cases that were known earlier, but derived with more pains:

1.) Let $G = \mathbb{SL}_2$ and H be the maximal unipotent subgroup consisting of upper triangular matrices. Then $k[G]^H$ is the coordinate algebra $k[V]$ of the affine plane $V = \mathbb{A}^2$, because H is the stabilizer G_x of the point $x = (1, 0)$ whose orbit $G \cdot x = \mathbb{A}^2 - \{0\}$ is dense and isomorphic to G/H and has a boundary of codimension 2. Here are two interesting applications of this situation:

a) Let A be the coordinate algebra $k[R_d]$ of the vector space R_d of binary forms of degree d. Then we get the isomorphism

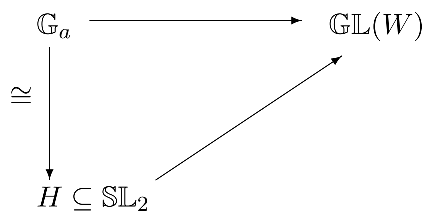
$$k[V \oplus R_d]^G \xrightarrow{\sim} k[R_d]^H$$

between 'covariants' and 'seminvariants', given by evaluating a covariant F at the point x,

$$F \longmapsto F((1,0),-),$$

where the image is the 'Leitglied' (leading term) of the covariant. This result goes back to Roberts (around 1870).

b) Let $\operatorname{char} k = 0$ and A be the coordinate algebra $k[W]$ of an arbitrary rational (finite dimensional) $\mathbb{G}_a$-module W. Then the representation of $\mathbb{G}_a$ extends to $\mathbb{SL}_2$ via the embedding by the Jordan normal form.



Therefore the invariant algebra $k[W]^H \cong k[V \oplus W]^G$ is finitely generated. This is Seshadri's proof [20] of Weitzenböck's theorem (1932). Fauntleroy recently found a proof of this theorem in positive characteristic, see these conference proceedings or [5][2]. The proof is a skillful elaboration of the given one in characteristic 0 but, strictly speeking, doesn't depend on Grosshans's principle.

2.) Somewhat more generally we can take G reductive and H, a maximal unipotent subgroup of G. The principle for this case was observed by several people, for the first time (in characteristic 0) by Hadžiev [10], see also [6] and [21].

3.) Now let $G = \mathbb{GL}_n$ act on the polynomial ring

$$k[X] = k[X_{ij} \mid 1 \leq i, j \leq n]$$

in a matrix of indeterminates by left translation. Let H be a subgroup of $\mathbb{SL}_n$ such that $k[X]^H$ is finitely generated. Then for any affine algebra A on

[2] This proof had a gap. Weitzenböck's theorem is still unproved in positive characteristic. In particular Corollary 3 is proved only in characteristic 0.

which $\mathbb{GL}_n$ (or a reductive group G between H and $\mathbb{GL}_n$) acts rationally, the invariant algebra A^H is finitely generated. This is a qualitative version of the old principle: 'If you know the invariants of n vectors, you know all invariants.' (Capelli 1887) – The n vectors are the columns of the n-by-n matrix X.

The *proof* is two lines:

$$A^H \cong (k[\mathbb{GL}_n]^H \otimes A)^{\mathbb{GL}_n},$$

(I interchanged left and right translation, but that doesn't matter) and

$$k[\mathbb{GL}_n]^H = k[X][1/\det]^H = k[X]^H[1/\det]$$

because $H \leq \mathbb{SL}_n$ and det is $\mathbb{SL}_n$-invariant.$\diamond$

Note that we need an action of a bigger reductive group G containing H – of course this is a disadvantage, but in view of Nagata's counterexample it even looks surprisingly good.

3 GROSSHANS SUBGROUPS

The following seems to be a **good substitute of Hilbert's 14th problem:**

> *Find the Grosshans subgroups of $\mathbb{GL}_n$ or more generally of a reductive group G.*

The formal **definition** of a **Grosshans subgroup** H of an affine algebraic group is: H is closed, G/H is quasiaffine, $k[G/H]$ is finitely generated. The technical condition 'G/H quasiaffine' is automatic if H is unipotent. Let me give 3 examples:

a) By Hadžievs result the maximal unipotent subgroups are Grosshans, even if G is not reductive.

b) The existence of non-Grosshans subgroups follows from Nagata's counterexample: There must be a situation

$$\mathbb{GL}_n \geq U \geq V, \; U \text{ and } V \text{ unipotent with } \dim U/V = 1,$$

such that U is Grosshans and V is not.

c) Generic stabilizers often are Grosshans subgroups. The following theorem generalizes a result by Grosshans [7]. Since some people recently were interested in it, I give the proof here.

THEOREM 1 *Let X be a factorial affine variety and G, an affine algebraic group acting on X. Then X has a dense open subset U such that the stabilizer G_x is a Grosshans subgroup of G for all $x \in U$.*

Remark. Instead of 'factorial' the following condition suffices: X is normal and each G-invariant divisor on X has finite order in the divisor class group $Cl(X)$, cf. [18].

Proof. I may assume G connected. There is a function $f \in k[X]$ such that the principal open subset X_f is G-stable and $k(X)^G$ is the quotient field of $k[X_f]^G$; this is well-known, cf. [13]. Choose functions $f_1, \ldots, f_n \in k[X_f]^G$ that generate the field $k(X)^G$. Let R be the algebra generated by $f_1, \ldots, f_n$ and Y be an affine model of R. Then $k(Y) = k(X)^G$, and the induced morphism $\pi : X_f \longrightarrow Y$ is dominant.

Now let $m = \max\{\dim G \cdot x \mid x \in X\}$ be the maximal orbit dimension. The set $Z = \{x \in X_f \mid \dim G \cdot x = m\}$ is G-stable and open dense in X, and $\dim Y = \dim X - m$. Since X_f is factorial, $X_f - Z = V(h) \cup A$ for a function $h \in k[X_f]$ with $\dim A \leq \dim X - 2$. Clearly X_{fh} is G-stable. Restricting π gives a dominant morphism $\sigma : X_{fh} \longrightarrow Y$. The fibres of σ are G-stable, and

$$\sigma^{-1}\sigma x = (\sigma^{-1}\sigma x \cap Z) \cup (\sigma^{-1}\sigma x \cap A) \text{ for all } x \in Z.$$

There is a dense open part $W \subseteq Z$ such that $\sigma^{-1}y$ has pure dimension m for all $y \in W$. Shrinking W we may assume that

$$\dim(\sigma^{-1}y \cap A) \leq m - 2 \text{ for all } y \in W.$$

Now $U = Z \cap \sigma^{-1}W$ is G-stable and open dense in X. Let $x \in U$. Then the closure $\overline{G \cdot x}$ in X_{fh} is an irreducible component of $\sigma^{-1}\sigma x$ – compare the dimensions. In $\overline{G \cdot x} - G \cdot x$ there is no $z \in Z$ since there is no room for the m-dimensional orbit $G \cdot z$. Thus $\overline{G \cdot x} - G \cdot x \subseteq \sigma^{-1}\sigma x \cap A$, and

$$\dim(\overline{G \cdot x} - G \cdot x) \leq \dim(\sigma^{-1}\sigma x \cap A) \leq m - 2 \leq \dim G \cdot x - 2.$$

Therefore G_x is a Grosshans subgroup of G.$\diamond$

There are some natural conjectures:

Conjecture 1 *(m) Each m-dimensional unipotent subgroup is Grosshans.*

This conjecture is false when $m = r^2 - 3$ and $r \geq 4$, and probably false when $m \geq 7$. Conjecture 1 (1) is true by Weitzenböck's theorem, and I guess this is the only positive case! Since nobody seems to have an approach to this problem, I make another conjecture:

Conjecture 2 *Each regular unipotent subgroup of a reductive group is Grosshans.*

'Regular' means 'normalized by a maximal torus', or, more concretely, given by a closed subset of the root system. For $\mathbb{GL}_n$ it means that the subgroup is defined by relations of the type $X_{ij} = 0$. The following example shows what this means:

$$\boxed{\begin{matrix} 1 & 0 & * & 0 \\ & 1 & 0 & * \\ & & 1 & 0 \\ & & & 1 \end{matrix}} = \{ \begin{pmatrix} 1 & 0 & a & 0 \\ & 1 & 0 & b \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \mid a, b \in k \text{ arbitrary}\}.$$

Such a pattern of zeroes and stars above the diagonal gives a subgroup of $\mathbb{GL}_n$, if and only if it is the incidence matrix of a strict ordering of the set $\{1, \ldots, n\}$.

Conjecture 2 is true for the unipotent radicals $H = R_u(P)$ of the parabolic subgroups P. This was shown by Hochschild and Mostow 1973 (for characteristic 0) [11], and by Grosshans 1983 (for the general case) [8]. Grosshans recently extended this result in several ways [9].

4 INVARIANT MINORS

My own contribution in [16], [17] is a large class of examples for $\mathbb{GL}_n$ – but unfortunately I have no general proof of conjecture 2, not even for $\mathbb{GL}_n$. My approach is the explicit determination of $k[X]^H$, where $X = (X_{ij})$ is the n-by-n matrix of indeterminates and $H \leq \mathbb{GL}_n$ regular unipotent. It looks promising because a lot of invariants are obvious: Consider a minor

$$\begin{vmatrix} X_{i_1 j_1} & \ldots & X_{i_1 j_m} \\ \vdots & & \vdots \\ X_{i_m j_1} & \ldots & X_{i_m j_m} \end{vmatrix},$$

shortly represented by the row $(i_1 \ldots i_m \mid j_1 \ldots j_m)$. When is it invariant? For the group at the end of section 3 we have

$$\begin{pmatrix} 1 & 0 & a & 0 \\ & 1 & 0 & b \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \cdot \begin{pmatrix} X_{11} & \ldots \\ X_{21} & \ldots \\ X_{31} & \ldots \\ X_{41} & \ldots \end{pmatrix} = \begin{pmatrix} X_{11} + aX_{31} & \ldots \\ X_{21} + bX_{41} & \ldots \\ X_{31} & \ldots \\ X_{41} & \ldots \end{pmatrix}.$$

So our minor is invariant if and only if the following is true:

- If it contains the row index 1, then it also contains 3,
- if it contains the row index 2, then it also contains 4.

This is because then H acts by elementary row operations. In general we read the condition off the constellation of stars:

- If the box representing the group contains stars at positions $l_1, \dots, l_m$ in the i-th row, then the minor has to contain the row indices $l_1, \dots, l_m$ along with i.

So we know the invariant minors. I would like to prove:

Conjecture 3 *The invariant algebra $k[X]^H$ is generated by the (finitely many) invariant minors.*

This would imply Conjecture 2 for $\mathbb{GL}_n$. In fact I can prove a much stronger result, but only for a large class of unipotent subgroups that however contains the unipotent radicals of the parabolics as simplest special cases. Since the proof (and even the statement of the result) uses some complicated combinatorial methods, I'll give only a very simple example that, of course, is not new.

Take $n = 2$ and H, the maximal unipotent subgroup consisting of upper triangular matrices. The invariant minors are:

$$(12 \,|\, 12) = \det, \quad (2 \,|\, 1) = X_{21}, \quad (2 \,|\, 2) = X_{22},$$

because, if we have the row index 1 we also must have the row index 2, so the minor is the full determinant. Now $k[X][1/X_{22}] = R[1/X_{22}][X_{12}]$, where R is the algebra generated by the invariant minors; H acts trivially on $R[1/X_{22}]$ and maps X_{12} to $X_{12} + sX_{22}$ with $s \in k$ arbitrary. Therefore

$$k[X][1/X_{22}]^H = R[1/X_{22}].$$

This kind of argument goes through for general n: There is a product ε of invariant minors such that $k[X][1/\varepsilon]^H = R[1/\varepsilon]$. This means that the analogue of Conjecture 3 for rational functions is true. However, as it is often the case, it is a major problem to get rid of this denominator.

Let me continue the example. We have

$$k[X]^H = k[X] \cap R[1/X_{22}] \supseteq R.$$

To get equality I have to show: If $f \in k[X]$ and $X_{22} f \in R$, then $f \in R$ – by induction I may assume $r = 1$. This is the hard core of the proof, that however is no problem for this example. Write a product of minors in the form of a bitableau, say

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 \\ 1 & & 1 & \\ 1 & & 2 & \\ 2 & & 1 & \end{array}\right).$$

These bitableaux span $k[X]$. When the columns increase, we have a 'standard' bitableau. The given one is not standard because of its last entry. The straightening law by Rota and others, see [3] for example, says (for the general case of n-by-n matrices):

The standard bitableaux are a basis of $k[X]$.

This generalizes some classical determinant identities. In the example we have

$$\left(\begin{array}{c|c} 1 & 2 \\ 2 & 1 \end{array}\right) = \left(\begin{array}{c|c} 1 & 1 \\ 2 & 2 \end{array}\right) - (12 \,|\, 12);$$

in a similar way each bitableau obviously is a linear combination of standard ones.

Now call a bitableau 'admissible', if each of its rows represents an invariant minor. Then R is spanned by the admissible bitableaux. In our example the admissible standard bitableaux are a basis of R: 'Admissible' means that there is no row $(1 \,|\, \ldots)$; but then the bitableau is already standard.

Now take $f \in k[X]$ such that $X_{22} f \in R$, and write it as a linear combination of standard bitableaux:

$$f = c_1 T_1 + \cdots + c_r T_r \ \ \text{(with nonzero coefficients } c_i\text{)}.$$

$$X_{22} f = \sum_{i=1}^{r} c_i T_i X_{22} \in R$$

is a linear combination of standard bitableaux $T_i X_{22}$. These have to be admissible, so have their parts T_i. Therefore $f \in R$.

In the general case, under suitable conditions on H, the proof goes the same way except that the minors of an n-by-n matrix behave a lot more complicated. This is where the nontrivial combinatorial techniques come in.

Finally let me note that conjecture 3 is true in the cases

- $\dim H \leq 3$,
- $n \leq 4$.

The first examples where the Grosshans property is unknown are

$$\boxed{\begin{array}{ccccc} 1 & 0 & * & * & 0 \\ & 1 & 0 & * & * \\ & & 1 & 0 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{array}} \quad \text{and} \quad \boxed{\begin{array}{ccccc} 1 & 0 & 0 & * & 0 \\ & 1 & 0 & * & * \\ & & 1 & 0 & * \\ & & & 1 & 0 \\ & & & & 1 \end{array}}$$

and this are the only exceptions for $n = 5$.

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