On Systems of Generators of Graded Algebras

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Let A be an $\mathbb{N}^l$-graded ring (commutative with 1),

$$A = \bigoplus_{\nu \in \mathbb{N}^l} A_\nu,$$

and let $k = A_0$. Consider the ideal

$$\mathfrak{m} = \bigoplus_{\nu \in \mathbb{N}^l - \{0\}} A_\nu.$$

(If k is a field, then $\mathfrak{m}$ is a maximal ideal, hence the notation.)

Lemma 1 *Let $h \in \mathfrak{m}^2$ be homogeneous of degree $\nu \in \mathbb{N}^l$. Then h is a finite sum*

$$h = \sum h' h''$$

where $h', h'' \in \mathfrak{m}$ are homogeneous of degree $< \nu$ in the natural (nonlinear) order of $\mathbb{N}^l$.

Proof. By definition of $\mathfrak{m}^2$ we may write h in the form

$$h = h'_1 h''_1 + \cdots + h'_r h''_r$$

with $h'_i, h''_i \in \mathfrak{m}$. Decompose all the h'_i and h''_i into their homogeneous parts—they are elements of $\mathfrak{m}$—and apply the distributive law. $\diamond$

In the following we use the natural homomorphism

$$\varphi : \mathfrak{m} \longrightarrow \mathfrak{m}/\mathfrak{m}^2$$

of A-modules.

Proposition 1 *Let $\mathcal{B} \subseteq \mathfrak{m}$. Then each of the following statements implies the following one:*

(i) $\mathcal{B}$ *generates the* k*-algebra* A*, i. e.* $A = k[\mathcal{B}]$.

(ii) $\mathcal{B}$ *generates the* A*-module* $\mathfrak{m}$*, i. e.* $\mathfrak{m} = A\mathcal{B}$.

(iii) $\varphi\mathcal{B}$ *generates* $\mathfrak{m}/\mathfrak{m}^2$ *as* k*-module.*

If all $b \in \mathcal{B}$ *are homogeneous, then these three statements are equivalent.*

Proof. "(i) $\Longrightarrow$ (ii)": Consider an arbitrary $f \in \mathfrak{m}$. Then f is a polynomial in finitely many $b_1, \ldots, b_r \in \mathcal{B}$ without constant term. Hence f is a linear coombination of monomials

$$g_s = b_1^{s_1} \cdots b_r^{s_r} \quad \text{where } s = (s_1, \ldots, s_r) > 0 \text{ in } \mathbb{N}^l.$$

Say $s_i > 0$ in $\mathbb{N}$. Then $g_s \in Ab_i \subseteq A\mathcal{B}$. Hence also $f \in A\mathcal{B}$.

"(ii) $\Longrightarrow$ (iii)": Let $f \in \mathfrak{m}$, thus

$$f = \sum_{i=1}^{r} g_i b_i \quad \text{with } g_i \in A \text{ and } b_i \in \mathcal{B}.$$

For all i, decompose $g_i = c_i + h_i$ with $c_i \in k$ and $h_i \in \mathfrak{m}$. Then

$$f = \sum_{i=1}^{r} c_i b_i + \underbrace{\sum_{i=1}^{r} h_i b_i}_{\in \mathfrak{m}^2},$$

$$\varphi f = \sum_{i=1}^{r} c_i \varphi b_i.$$

"(iii) $\Longrightarrow$ (i)": Here we assume that all $b \in \mathcal{B}$ are homogeneous. We proceed by induction over the degree ν, and have to show that $A_\nu \subseteq k[\mathcal{B}]$. Note that this claim is trivial for $\nu = 0$ since $A_0 = k$.

Now assume that $\nu \neq 0$, and $A_\mu \subseteq k[\mathcal{B}]$ for all $\mu < \nu$. Take an arbitrary $f \in A_\nu$. In particular $f \in \mathfrak{m}$, and by (iii)

$$\varphi f = \sum_{i=1}^{r} a_i \varphi b_i \quad \text{with } a_i \in k \text{ and } b_i \in \mathcal{B}.$$

This implies

$$f = \sum_{i=1}^{r} a_i b_i + h \quad \text{with } h \in \mathfrak{m}^2.$$

Consider a homogeneous part h_μ of h of degree $\mu \neq \nu$. Then $0 = \sum' a_i b_i + h_\mu$ where the sum is over the i with $\deg b_i = \mu$. (Here we use that all b_i are homogeneous.) Hence $h_\mu \in k[\mathcal{B}]$.

Now for the homogeneous part h_ν. Since $h_\nu \in \mathfrak{m}^2$ the lemma provides a finite sum $h_\nu = \sum h'_\nu h''_\nu$ with homogeneous elements $h'_\nu, h''_\nu \in \mathfrak{m}$, necessarily of degree $< \nu$, hence $\in k[\mathcal{B}]$ by induction. Taken together this implies $f \in k[\mathcal{B}]$. $\diamond$

In the most important special case k is a field and $\mathfrak{m}/\mathfrak{m}^2$ is a finite-dimensional vector space over k, say of dimension n. Then the proposition says that each system $\mathcal{B}$ of generators of the k algebra A contains at least n elements since $\varphi\mathcal{B}$ spans $\mathfrak{m}/\mathfrak{m}^2$. Now consider the set $\mathcal{B}'$ of all the homogeneous parts of all elements of $\mathcal{B}$. Then also $\mathcal{B}'$ generates the k-algebra A. Choose a subset $\mathcal{B}'' \subseteq \mathcal{B}'$ such that $\varphi\mathcal{B}''$ is a basis of $\mathfrak{m}/\mathfrak{m}^2$. Then by the proposition also $\mathcal{B}''$ generates the k-algebra A. This consideration proves

Corollary 1 *Let k be a field and A be an $\mathbb{N}^l$-graded k-algebra with $A_0 = k$. Assume $\dim_k \mathfrak{m}/\mathfrak{m}^2 = n$ is finite. Then the minimal cardinality of a set of generators of A is n, and there is a set of* homogeneous *generators of cardinality n. Moreover each minimal set of homogeneous generators contains exactly n elements.*