The Morozov-Jacobson Theorem on 3-dimensional Simple Lie Subalgebras

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The Morozov-Jacobson theorem says that every nilpotent element of a semisimple Lie algebra is contained in a 3-dimensional simple subalgebra, i. e. a subalgebra isomorphic with $\mathfrak{sl}_2$. Morozov in [6] stated the theorem for the base field of complex numbers. His proof however contained a gap that Jacobson filled in [4]. Jacobson more generally formulated the proof for a base field of characteristic 0. In fact the proof is valid also if the characteristic of the base field is "large enough", depending on the nilpotent element under consideration, and if the Lie algebra is the Lie algebra of a semisimple algebraic group.

The Morozov-Jacobson theorem is basic for the classification of nilpotent elements in semisimple Lie algebras and unipotent elements in semisimple algebraic groups, see [5], [1], and [10, III.3.29].

In this note—that is an excerpt from [7]—we remove the restriction on the characteristic up to very small exceptions: The characteristic should be $\neq 2$ and good for the algebraic group, that means $\neq 3$ if the group contains a component of exceptional type, and $\neq 5$ if the group contains a component of exceptional type $\mathbf{E}_8$. Because the proof uses an invariant scalar product on the Lie algebra, it depends for some small characteristics on the classification of semisimple algebraic groups.

1 $\mathfrak{sl}_2$-Triples

For algebraic groups we use standard notation as in [3]. Let k be an algebraically closed field and G be a reductive algebraic group over k with Lie algebra $\mathfrak{g}$. A triple (h, x, y) of elements of $\mathfrak{g}$ is called an **$\mathfrak{sl}_2$-triple** if h is semisimple, x and y are nilpotent, and

$$[hx] = 2x, \quad [hy] = -2y, \quad [xy] = h,$$

i. e. h, x, y span a subalgebra isomorphic with $\mathfrak{sl}_2$, an isomorphism being

$$h \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad x \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Note that $\mathfrak{sl}_2$ is a simple Lie algebra if and only if $\operatorname{char} k \neq 2$. Also note that (in characteristic $\neq 2$) (h, x, y) is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ if and only if it is not the all-zero triple and each of its components in the direct decomposition of $\mathfrak{g}$ corresponding to the almost simple components of G is an $\mathfrak{sl}_2$-triple or $(0,0,0)$. Therefore we may without restriction switch to the case of almost simple G whenever convenient.

2 One-Parameter Subgroups and Gradings

Let G be reductive and T be a maximal torus of G. A one-parameter subgroup (OPSG) $\lambda\colon \mathbb{G}_m \longrightarrow T$ is characterised by integers $\langle\alpha,\lambda\rangle$ for all all $\alpha \in \Delta$, a basis of the root system $\Phi = \Phi(T,G)$. These integers are given by

$$\operatorname{Ad}\lambda(c)\cdot x = c^{\langle\alpha,\lambda\rangle}x \text{ for all } c \in \mathbb{G}_m \text{ and } x \in \mathfrak{g}_\alpha.$$

A OPSG of G induces a grading of the Lie algebra $\mathfrak{g}$ of G as follows:

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i \quad \text{where } \operatorname{Ad}\lambda(c)\cdot x = c^i x \text{ for all } c \in \mathbb{G}_m \text{ and } x \in \mathfrak{g}_i.$$

Note that in prime characteristic $\mathfrak{g}$ might have gradings that are not induced by a OPSG.

Let $G_0 = Z_G(\lambda(\mathbb{G}_m))^0$. Then $\mathfrak{g}_0$ is the Lie algebra of G_0, and each $\mathfrak{g}_i$ is a G_0-submodule of $\mathfrak{g}$. If we choose a maximal torus $T \supseteq \lambda(\mathbb{G}_m)$, then $\mathfrak{g}_i$ is the sum of all root spaces $\mathfrak{g}_\alpha$ where $\langle\alpha,\lambda\rangle = i$.

Now let $x \in \mathfrak{g}$ be nilpotent. The normalizer $N := N_G(kx)$ consists of all group elements that map the one-dimensional subspace kx into itself, and it contains the centralizer $Z := G_x = Z_G(x)$. Because Z is the kernel of the action of N on kx, we have $\dim N/Z \le 1$. This leaves two possibilities:

- N/Z is finite. Then each nilpotent class meets kx only in finitely many points, therefore infinitely many nilpotent classes meet kx. This could occur only if $\mathfrak{g}$ had infinitely many nilpotent classes. This never occurs, see [2]. But the proof is simple only if the characteristic is good for G, see [8], or [10].
- $\dim N/Z = 1$. Then the connected component $(N/Z)^0$ is $\cong \mathbb{G}_m$ or $\cong \mathbb{G}_a$. Since the only 1-dimensional representation of the additive group $\mathbb{G}_a$ is trivial, $(N/Z)^0 \cong \mathbb{G}_m$, and therefore N contains a 1-dimensional torus that acts transitively on the punctured line $k^\times x$.

This consideration proves:

Proposition 1 *Let G be reductive, and let $x \in \mathfrak{g}$ be nilpotent. Assume that the number of nilpotent classes in $\mathfrak{g}$ is finite. Then there is a OPSG $\lambda\colon \mathbb{G}_m \longrightarrow G$ and an integer $i \in \mathbb{Z} - \{0\}$ such that $x \in \mathfrak{g}_i$ for the induced grading.*

For a stronger version of Proposition 1 see [9].

Corollary 1 *The image of λ may be chosen in the semisimple part (G,G).*

Proof. The proof of Proposition 1 works for (G,G) as well as for G. $\diamond$

3 Invariant Scalar Products

Now we assume that $\operatorname{char} k \neq 2$ and that there is a G-invariant scalar product $(\bullet|\bullet)$ (i. e. nondegenerate symmetric bilinear form) on $\mathfrak{g}$. This is fulfilled if G has no component of type $\mathbf{A}_l$ and the characteristic is good for G ([8] or [10, III.4.1]), or if $G \cong \mathbf{GL}_n$. In particular $(\bullet|\bullet)$ is associative: $([xy]|z) = (x|[yz])$ for all $x, y, z \in \mathfrak{g}$.

Lemma 1 $\mathfrak{g}_x = [x\mathfrak{g}]^\perp$ *for all* $x \in \mathfrak{g}$.

Proof. We have $\mathfrak{g}_x = \{z \in \mathfrak{g} \mid [zx] = 0\}$ and $z \in [x\mathfrak{g}]^\perp \iff (z|[xy]) = 0$ for all $y \in \mathfrak{g} \iff ([zx]|y) = 0$ for all $y \in \mathfrak{g} \iff [zx] = 0$. $\diamond$

Let T be a maximal torus of G, $\mathfrak{t} \subseteq \mathfrak{g}$ its Lie algebra, Φ the corresponding root system, and

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$$

the root space decomposition. Then $\mathfrak{t}$ is a nondegenerate subspace and

$$\mathfrak{t}^\perp = \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha.$$

Furthermore $\mathfrak{g}_\alpha \perp \mathfrak{g}_\beta$ except for $\beta = -\alpha$, and $\mathfrak{g}_{-\alpha}$ is the dual subspace of $\mathfrak{g}_\alpha$ (i. e. $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ is a nondegenerate subspace).

Now let $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ ba a grading given by a OPSG $\lambda\colon \mathbb{G}_m \longrightarrow T$. Because the $\mathfrak{g}_i$ are sums of root spaces we immediately conclude that $\mathfrak{g}_0$ is a nondegenerated subspace and $\mathfrak{g}_{-i}$ is the dual space of $\mathfrak{g}_i$. Let $\mathfrak{s}$ be the Lie algebra of $S := \lambda(\mathbb{G}_m)$.

Lemma 2 *For* $x \in \mathfrak{g}_i$*:*

(i) $\mathfrak{g}_x^\perp \cap \mathfrak{g}_0 = [x\mathfrak{g}_{-i}]$.

(ii) $\mathfrak{g}_{0x}^\perp \cap \mathfrak{g}_0 = \mathfrak{g}_x^\perp \cap \mathfrak{g}_0$.

(iii) *If* $\mathfrak{g}_{0x} \perp \mathfrak{s}$*, then* $\mathfrak{s} \subseteq [x\mathfrak{g}_{-i}]$.

Proof. (i) By Lemma 1 we have $\mathfrak{g}_x^\perp = [x\mathfrak{g}]$. Since $\mathfrak{g} = \bigoplus_j \mathfrak{g}_j$ and x is homogeneous, we have $[x\mathfrak{g}] = \bigoplus_j [x\mathfrak{g}_j]$, and $[x\mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$. Therefore $\mathfrak{g}_x^\perp \cap \mathfrak{g}_0 = [x\mathfrak{g}] \cap \mathfrak{g}_0 = [x\mathfrak{g}_{-i}]$.

(ii) $\mathfrak{g}_{0x}^\perp \supseteq \mathfrak{g}_x^\perp$ because $\mathfrak{g}_{0x} \subseteq \mathfrak{g}_x$. For $y \in \mathfrak{g}_{0x}^\perp \cap \mathfrak{g}_0$ and $z \in \mathfrak{g}_x$ we have $(y|z) = \sum (y|z_j) = (y|z_0) = 0$, where z_j is the $\mathfrak{g}_j$-component of z. Therefore $\mathfrak{g}_{0x}^\perp \cap \mathfrak{g}_0 \subseteq \mathfrak{g}_x^\perp$.

(iii) If $\mathfrak{s} \subseteq \mathfrak{g}_{0x}^\perp$, then $\mathfrak{s} \subseteq \mathfrak{g}_{0x}^\perp \cap \mathfrak{g}_0 = [x\mathfrak{g}_{-i}]$ by (ii) and (i). $\diamond$

Proposition 2 *For* $x \in \mathfrak{g}_i$ *with* $\mathfrak{g}_{0x} \perp \mathfrak{s}$ *there is an* $\mathfrak{sl}_2$*-triple* (h, x, y) *with* $h \in \mathfrak{s}$ *and* $y \in \mathfrak{g}_{-i}$.

Proof. $\mathfrak{s}$ acts on $\mathfrak{g}_i$ by multiplication with scalars, and the action is non-trivial. Therefore there is an $h \in \mathfrak{s}$ with $[hx] = 2x$. From Lemma 2 (iii) we get a $y \in \mathfrak{g}_{-i}$ such that $[xy] = h$. Then $[hy] = ty$ with some $t \in k$. The chain $th = t[xy] = [x[hy]] = [h[xy]] + [y[hx]] = [hh] + 2[yx] = -2h$ shows that $t = -2$. ◇

4 Distinguished Nilpotent Elements

For the moment we drop the assumption on char k. A nilpotent element $x \in \mathfrak{g}$ is called **distinguished** if a maximal torus of G_x is contained in the center of G; for semisimple G this means that G_x^0 is a unipotent subgroup of G. A Levi-type subgroup of G is the centralizer of a subtorus of G or, in other words, a Levi factor of a parabolic subgroup.

Proposition 3 *Let G be reductive. Let $x \in \mathfrak{g}$ be nilpotent. Then there is a Levi-type subgroup H of G such that x is a distinguished nilpotent element of $\mathfrak{h} = \mathrm{Lie}(H)$.*

Proof. Let S be a maximal torus of G_x. Then $H := Z_G(S)^0$ is a Levi-type subgroup with Lie algebra $\mathfrak{h} = \mathfrak{z}_{\mathfrak{g}}(S)$. Therefore $x \in \mathfrak{h}$. By definition of H the nilpotent element x is distinguished in $\mathfrak{h}$. ◇

Proposition 4 *Let G be reductive in good characteristic $\neq 2$. Let $\mathfrak{g}$ be graded by a OPSG $\lambda\colon \mathbb{G}_m \longrightarrow (G, G)$. Let $h \in \mathfrak{s} := \mathrm{Lie}(\lambda(\mathbb{G}_m))$ be the element that acts on $\mathfrak{g}_i$ by multiplication with 2. Let $x \in \mathfrak{g}_i$ be distinguished nilpotent. Then there is a $y \in \mathfrak{g}_{-i}$ such that (h, x, y) is an $\mathfrak{sl}_2$-triple.*

Proof. If G has no normal subgroup of type $\mathbf{A}_l$, this follows from Proposition 2: Since $\mathfrak{g}_{0x} \subseteq \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{u}$, where $\mathfrak{u}$ is the Lie algebra of a maximal unipotent subgroup of G_0, we have $\mathfrak{g}_{0x} \perp \mathfrak{s}$.

For the general case we yet have to consider the case G of type $\mathbf{A}_l$, where we may replace G by $\mathbf{GL}_n$ and again apply Proposition 2. ◇

Theorem 1 (Morozov-Jacobson) *Let G be semisimple in good characteristic $\neq 2$, and let $x \in \mathfrak{g}$ be nilpotent. Then there are $h, y \in \mathfrak{g}$ such that (h, x, y) is an $\mathfrak{sl}_2$-triple.*

Proof. By Proposition 3 we find a reductive subgroup H of G such that x is distinguished in the Lie algebra $\mathfrak{h}$ of H. By Proposition 1 we find a OPSG of H such that x is homogenous for the corresponding grading. By Proposition 4 we find an $\mathfrak{sl}_2$-triple (h, x, y) in $\mathfrak{h}$, a forteriori in $\mathfrak{g}$. ◇

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