Remarks on Subset Sums

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Consider an r-element subset T of an (additively written) abelian group (or $\mathbb{Z}$-module) M. For each of the different 2^r subsets $U \subseteq T$ we can form the **subset sum**

$$\Sigma(U) := \sum_{t \in U} t,$$

in particular $\Sigma(\emptyset) = 0$. We denote the set of subset sums for $T \subseteq M$ by

$$\mathcal{S}(T) = \{\Sigma(U) \mid U \subseteq T\}.$$

and may ask:

- How many different values can the subset sums $\Sigma(U)$ for $U \subseteq T$ take? In other words, how large is $\Delta(T) = \#\mathcal{S}(T)$? An obvious lower bound is r (or $r+1$ if $0 \notin T$), an obvious upper bound is 2^r.
- Do the subset sums of T cover M, in other words, is $\mathcal{S}(T) = M$?
- Is $0 = \Sigma(U)$ for some *nonvoid* subset $U \subseteq T$? What about such "zero-sum" subsets U? How many minimal ones exist? How large can they be?
- What about T if no nontrivial subset sum $\Sigma(U)$, $U \subseteq T$, vanishes? How large can such a "zerofree" set T be?

1 Elementary Examples

Example 1 If M is a vector space over some field and the $t \in T$ are linearly independent, then all 2^r subset sums are different.

Example 2 In the case $M = \mathbb{Z}$ and $T = \{1, \ldots, r\}$ we certainly have

$$0 \;\leq\; \Sigma(U) \;\leq\; \Sigma(T) = \sum_{i=1}^{r} i \;=\; \frac{r \cdot (r+1)}{2}$$

for $U \subseteq T$, and $\Sigma(U)$ is an integer. This bounds the number of different subset sums to $1 + r(r+1)/2$.

It is easy to show that the righthand side of Example 2 provides a general lower bound on the number of different sum values if T consists of positive real numbers.

Theorem 1 *If T is an r-element set of positive real numbers, then the number of different values $\Sigma(U)$ for $U \subseteq T$ is at least*

$$1 + \frac{r \cdot (r+1)}{2} .$$

Proof. Let $T = \{t_1, \ldots, t_r\}$ where $0 < t_1 < \ldots < t_r$. Then the subset sums

$$0,\ t_1, \ldots, t_r,\ t_r + t_1, \ldots, t_r + t_{r-1},$$

$$t_r + t_{r-1} + t_1, \ldots, t_r + t_{r-1} + t_{r-2},$$

$$\ldots,\ t_r + \cdots + t_1$$

form a strictly increasing chain of $1 + r + (r-1) + (r-2) + \cdots + 1$ real numbers. $\diamond$

Corollary 1 *For $T = \{1, \ldots, r\}$ the set $\mathcal{S}(T)$ consists exactly of the integers $1, \ldots, r\,(r+1)/2$.*

2 Zerofree Subsets

Let M be an abelian group of order $m \leq \infty$ and $T \subseteq M - \{0\}$ a finite subset with $r := \#T$. We start with some examples, elementary observations, and a definition.

Example 1, $r = 2$. Then $T = \{a, b\}$ with $b \neq a$, and we distinguish two cases:

- $a + b = 0$: Then there are exactly 3 subset sums of T, the sums 0, a, b.
- $a + b \neq 0$: Then $a + b \neq 0, a, b$. Therefore we have exactly 4 different subset sums.

Remark The number of different subset sums is bounded by m (relevant only if m is finite).

Example 2, $M = \mathbb{Z}/m\mathbb{Z}$, $1 \leq r \leq m-1$, and $T = \{1, \ldots, r\}$. From Section 1 we know that the subset sums of T (considered as integers) take all values in the integer interval $[0 \ldots R]$ where $R = r\,(r+1)/2$.

- If $m \leq R+1$, then the subset sums $\bmod m$ take all possible values in $0 \ldots m-1$.
- If $m \geq R+1$, then the subset sums $\bmod m$ take all possible values in $0 \ldots R$.

Special cases

- $r=3$, $R=6$, $m \geq 7$: The possible values $\bmod m$ are $0, \ldots, 6$, and their number is $7 = 2r+1$.
- $r=4$, $R=10$, $m \geq 11$: The possible values $\bmod m$ are $0, \ldots, 10$, and their number is $11 = 2r+3$.

Definition Call a subset $T \subseteq M$ **zerofree** if no subset sum $\Sigma(U)$, $U \subseteq T$, $U \neq \emptyset$, is 0. (In particular then $0 \notin T$.) The integer

$$\mathsf{zf}(M) = \max\{\#T \mid T \subseteq M \text{ zerofree}\}$$

is the **zerofree bound** of M.

Call **diversity** $\Delta(T) = \#\mathcal{S}(T)$ the number of different subset sums $\Sigma(U)$ where $U \subseteq T$ (no matter whether T is zerofree or not).

Call r-**diversity** of M the minimum

$$\Delta_M(r) = \min\{\Delta(T) \mid T \subseteq M \text{ zerofree with } \#T = r\}.$$

(For convenience set the minimum over an empty set to ∞.)

Remark 1 For very small r we have the obvious statements:

- $\Delta(\emptyset) = 1$. (The set $\emptyset$ is zerofree.) Hence $\Delta_M(0) = 1$.
- If $\#T = 1$, then T is zerofree if and only if $0 \notin T$, and then $\Delta(T) = 2$. Hence $\Delta_M(1) = 2$.
- By example 2, if T is zerofree and $\#T = 2$, then $\Delta(T) = 4$. Hence $\Delta_M(2) = 4$.

Remark 2 Recall the obvious bounds $r+1 \leq \Delta(T) \leq \min(2^r, m)$ for zerofree T, hence $r+1 \leq \Delta_M(r) \leq \min(2^r, m)$.

We restate the result of example 2:

Lemma 1 *For $M = \mathbb{Z}/m\mathbb{Z}$:*

(i) *If $m \leq r\,(r+1)/2$, then $\Delta(\{1, \ldots, r\}) = m$.*

(ii) *If $m \geq 1 + r\,(r+1)/2$, then $\Delta(\{1, \ldots, r\}) = 1 + r\,(r+1)/2$.*

Some immediate observations:

Lemma 2

$$\mathsf{zf}(\mathbb{Z}/m\mathbb{Z}) \geq \max\left\{r \;\middle|\; \frac{r\,(r+1)}{2} < m\right\} = \left\lceil \sqrt{2m+\frac{1}{4}} - \frac{3}{2} \right\rceil.$$

Proof. The inequality follws from the zerofreeness of $\{1,\ldots,r\}$. For the equality we use the formula for the roots of the quadratic polynomial x^2+x-2m:

$$x = -\frac{1}{2} \pm \sqrt{2m+\frac{1}{4}}$$

Then (for $r > 0$)

$$\frac{r\,(r+1)}{2} < m \Longleftrightarrow r^2+r < 2m \Longleftrightarrow r < \left\lceil \sqrt{2m+\frac{1}{4}} - \frac{1}{2} \right\rceil \Longleftrightarrow r \leq \left\lceil \sqrt{2m+\frac{1}{4}} - \frac{3}{2} \right\rceil.$$

◇

Table 1 shows some results, calculated by the Python (or SageMath) program in Appendix C. We see that our lower bound $q(m)$ is already close to the truth. As we'll see in Section 6 for m prime $q(m)$ always *equals* the true value (Theorem of Olson/Balandraud).

m	2	3	4	5	6	7	8	9	10	11	12
$q(m)$	1	1	2	2	2	3	3	3	3	4	4
$\mathsf{zf}(m)$	1	1	2	2	3	3	3	4	4	4	4

Table 1: Zerofree bound of $\mathbb{Z}/m\mathbb{Z}$ where q is the lower bound from Lemma 2.

Lemma 3 *Let M_2 be the subgroup of 2-torsion elements (the $a \in M$ with $2a = 0$) and $m_2 = \#M_2$. Let T be zerofree. Then $\#T \leq (m+m_2)/2 - 1$.*

Proof. From each pair $(a, -a)$ with $a \notin M_2$ only one partner can be in T. This makes at most $(m-m_2)/2$ elements. Moreover T may contain the elements of M_2 except 0. ◇

Corollary 1 *If $M = \mathbb{Z}/m\mathbb{Z}$ and $T \subseteq M$ is zerofree, then $\#T \leq m/2$.*

Proof. $m_2 = 1$ if m is odd, and $m_2 = 2$ if m is even. ◇

Corollary 2 $\mathsf{zf}(\mathbb{Z}/m\mathbb{Z}) \leq m/2$.

Lemma 4 *If T is zerofree, then $\Sigma(T) \neq \Sigma(S)$ for any proper subset $S \subset T$. More generally $\Sigma(U) \neq \Sigma(S)$ for two subsets $S \subset U \subseteq T$.*

Proof. Otherwise $\Sigma(U - S) = 0$ for the nonempty subset $U - S \subseteq T$. ◇

Lemma 5 *If a subset $S \subseteq T$ is not zerofree, then T itself is not zerofree.*

3 Zerofree Sets with Three Elements

Lemma 6 *Let M be an abelian group and $T = \{t_1, t_2, t_3\} \subseteq M$ be zerofree.*

(i) *The five subset sums 0, t_1, t_2, t_3, $t_1 + t_2 + t_3$ are different.*

(ii) *Assume $\{i, j, k\} = \{1, 2, 3\}$. Then the sum $t_i + t_j$ equals some other subset sum of T if and only if $t_i + t_j = t_k$.*

(iii) *$\Delta(T) = 8 - s$ where s is the number of true equations in the system*

$$t_1 + t_2 \stackrel{?}{=} t_3 \tag{1}$$

$$t_1 + t_3 \stackrel{?}{=} t_2 \tag{2}$$

$$t_2 + t_3 \stackrel{?}{=} t_1 \tag{3}$$

(iv) *$\Delta(T) \geq 6$.*

Proof. (i) is trivial.

(ii) The subset sum $t_i + t_j$ is different from 0, t_i, t_j, $t_i + t_k$, $t_j + t_k$, and $t_i + t_j + t_k = \Sigma(T)$. The only remaining possibility is $t_i + t_j = t_k$.

(iii) By (ii) the equations (1)–(3) describe the only way a two-element sum might equal any of the other subset sums.

(iv) Assume $\Delta(T) < 6$. Then all three equations (1)–(3) are true. Adding (1) and (2) yields $2\,t_1 = 0$, hence $t_1 = -t_1$. Then (3) yields $t_2 + t_3 = -t_1$, hence the contradiction $\Sigma(T) = 0$. $\diamond$

Lemma 7 *Let $t \in M$ have order 2, $u \in M$ have order > 2, and $2u \neq t$. Let $T = \{t, u, t + u\}$. Then*

(i) *$\#T = 3$.*

(ii) *T is zerofree.*

(iii) *$\Delta(T) = 6$.*

Proof. (i) is trivial.

(ii) The eight subset sums $\Sigma(U)$ are

$$0,\ t,\ u,\ t+u,\ t+u,\ 2t+u = u,\ t+2u,\ 2t+2u = 2u. \tag{4}$$

Except for $U = \emptyset$ they are $\neq 0$—the only case in doubt might be $\Sigma(\{u, t+u\}) = t + 2u$ whose nonvanishing is granted by the definition of T, $2u \neq t = -t$.

(iii) The first four sums in (4) represent different elements of M.

The sum $t + 2u$ is different from 0 (as shown in (ii)), and from $t + u$ and t. It is also different from u since $t + 2u = u$ contradicts $u \neq t = -t$.

The last sum $2u$ is different from 0, t, u, $t+u$, and $t+2u$.

Therefore the eight sums in (4) represent exactly six different elements of M. $\diamond$

Proposition 1 *Let M be an abelian group and $T \subseteq M$ be zerofree with $\#T = 3$.*

(i) *Assume T contains no element of order 2. Then $\Delta(T) \geq 7$.*

(ii) *If $\Delta(T) = 6$, then T contains an element t of order 2, an element u of order > 2 with $2u \neq t$, and $T = \{t, u, t+u\}$.*

Proof. By Lemma 6 we have $\Delta(T) \geq 6$, and in the case of equality exactly two of the equations (1)–(3) must be true. Without loss of generality we may assume that (1) and (2) are true. Then

$$t_1 + t_2 = t_3 = t_2 - t_1, \quad \text{hence } 2\,t_1 = 0.$$

Under the assumption of (i) this is a contradiction, hence $\Delta(T) \neq 6$.

(ii) If $\Delta(T) = 6$, then t_1 has order 2, and $T = \{t_1, t_2, t_1 + t_2\}$. Since (3) is false, $2\,t_2 \neq 0$. Finally $2\,t_2 = t_1$ would make the subset sum $t_2 + (t_1 + t_2)$ zero, contradiction. $\diamond$

In the case $M = \mathbb{Z}/m\mathbb{Z}$ the exceptional sets from Lemma 7 exist only if m is even and have the form

$$T(m,a) := \{a, \frac{m}{2}, \frac{m}{2} + a\} \quad \text{for } 1 \leq a < \frac{m}{2},\ a \neq \frac{m}{4}\,.$$

Corollary 1 *Let $m \geq 6$ and $T \subseteq \mathbb{Z}/m\mathbb{Z}$ be zerofree with $\#T = 3$. Then the following statements are equivalent:*

(i) $\Delta(T) = 6$.

(ii) *m is even, and $T = T(m,a)$ for some a with $1 \leq a < m/2$, $a \neq \frac{m}{4}$.*

Corollary 2 *The number of zerofree subsets $T \subseteq \mathbb{Z}/m\mathbb{Z}$ with $\#T = 3$ and $\Delta(T) = 6$ is*

$$\begin{cases} 0 & \text{if } m \text{ is odd or } m \leq 5, \\ \frac{m}{2} - 2 & \text{if } m \text{ is even and } m \equiv 0 \pmod 4, \\ \frac{m}{2} - 1 & \text{if } m \text{ is even and } m \equiv 2 \pmod 4. \end{cases}$$

4 The Moser-Scherk Theorem

As a preliminary consideration for larger subsets we ask how many different two-element sums can be formed. A fundamental result was proved by SCHERK as a solution to a problem posed by MOSER, see [8].

Theorem 2 *Let M be an abelian group, $A, B \subseteq M$ finite subsets with* $0 \in A$ *and* $0 \in B$*. Assume that $a + b = 0$ for $a \in A$ and $b \in B$ only if $a = b = 0$. Then*

$$\#(A + B) \geq \#A + \#B - 1.$$

Remarks.

1. A and B need not to be disjoint.
2. In the special case $B = A$ we have $\#(A + A) \geq 2 \cdot \#A - 1$.
3. $A \subseteq A + B$ since $0 \in B$, and $B \subseteq A + B$ since $0 \in A$.

Proof. Induction on $n = \#B$. The case $n = 1$ ist trivial since $B = \{0\}$, $A + B = A$.

Now assume that $n = \#B \geq 2$. Take $b_0 \in B - \{0\}$. Since $0 \notin A + b_0$, but $0 \in A$, and $\#(A + b_0) = A$, there is an $a_0 \in A$ with $a_0 + b_0 \notin A$. Therefore the subset

$$Y := \{y \in B \mid a_0 + y \notin A\}$$

is nonvoid and proper (since $0 \in B - Y$). Let $X = a_0 + Y \subseteq M$. Then

$$0 < \#X = \#Y < \#B.$$

The construction of Y implies that X and A are disjoint. If we set

$$A' := A \cup X, \quad B' := B - Y,$$

then we have $\#A' = \#A + \#X$, $\#B' = \#B - \#X \leq n - 1$, and $0 \in A'$, $0 \in B'$. Claim:

(i) $A' + B' \subseteq A + B$.

(ii) If $c + d = 0$ for $c \in A'$ and $d \in B'$, then $c = d = 0$.

For the proof we take $c \in A'$ and $d \in B'$.

Case 1, $c \in A$. Then $c + d \in A + B$, and if $c + d = 0$, then $c = d = 0$.

Case 2, $c \in A' - A$, in particular $c \neq 0$. Then $c \in X$, thus $c = a_0 + y$ with $y \in Y$,

$$c + d = (a_0 + y) + d = \underbrace{(a_0 + d)}_{\in A} + y \in A + B$$

since $d \in B - Y$, and $c + d = 0$ implies that $a_0 + d = y = 0$, contradiction.

Now we may apply the induction hypothesis and get

$$\#(A + B) \geq \#(A' + B') \geq \#A' + \#B' - 1 = \#A + \#B - 1.$$

◇

Note There are a number of similar results in the literature, the most prominent ones being:

- The CAUCHY-DAVENPORT theorem: Let $A, B \subseteq M = \mathbb{Z}/p\mathbb{Z}$ (p prime) with $A+B \neq M$. Then $\#(A+B) \geq \#A+\#B-1$ (as in the MOSER-SCHERK theorem but without its restricting assumptions on A and B).
- KNESER's addition theorem: Let $A, B \subseteq M$ be finite nonempty subsets. Then there exists a subgroup $H \subseteq M$ with $A+B+H = A+B$ (i. e. consisting of "periods" of $A+B$) such that $\#(A+B) \geq \#(A+H) + \#(B+H) - \#H$.
- The ERDŐS-HEILBRONN conjecture, proved by HAMIDOUNE and DIAS DA SILVA: Let $M = \mathbb{Z}/p\mathbb{Z}$, an $A \subseteq M$ be an r-element subset. Then $\#(A\dot{+}A) \geq \min\{p, 2r-3\}$ where $A\dot{+}A$ denotes the set of sums $a+b$ for $a, b \in A$ with $a \neq b$.

5 The Eggleton-Erdős Theorem

Lemma 8 *Let M be an abelian group and $T \subseteq M$ an r-element subset with $r \geq 2$. Then*

$$\Delta(T) \geq \min\{\Delta_M(r-1) + 2, 2r+1\}.$$

Proof. First assume the existence of a $u \in T$ that is not a subsum of $T_u := T - \{u\}$. Let $S := \{\Sigma(U) \mid U \subseteq T_u\}$. Then $u \notin S$, and $N := \#S \geq \Delta_M(r-1)$. Also the full sum $\Sigma(T)$ is not in S, for otherwise $\Sigma(T) = \Sigma(U)$ with some $U \subseteq T_u$, contradicting Lemma 4. Since $u \neq \Sigma(T)$ we found two additional elements, hence $\Delta(T) \geq N+2$.

We are left with the case where each $u \in T$ is a subset sum of $T - \{u\}$. Applying Theorem 2 to $A = B = T \cup \{0\}$ with $\#A = r+1$ yields $\#(A+A) \geq 2\,(r+1) - 1 = 2r+1$. We want to show that each element of $A+A$ is a subset sum of T. For $t+u$ with different elements $t, u \in T$ this is trivial, likewise if one or two of the summands are zero. But what if we add two identical elements of T? For $u \in T$ we have

$$u = \sum_{t \neq u} \varepsilon_t\, t \quad \text{where } \varepsilon_t = 0 \text{ or } 1, \text{ not all} = 0.$$

Hence $u + u = u + \sum \varepsilon_t\, t$, a subsum of T. We conclude that $\Delta(T) \geq \#(A+A) \geq 2r+1$. ◇

Theorem 3 *Let M be an abelian group and $r \geq 1$. Then*

(i) $\Delta_M(r) \geq 2r$.

(ii) $\Delta_M(r) \geq 2r+1$ *if* $r \geq 4$.

Proof. We reason by induction on r.

(i) is true for $r = 1$ by Remark 1 in Section 2, and the induction step is provided by Lemma 8.

(ii) is proved the same way if we start the induction at $r = 4$ and use the following Lemma 9. ◇

Lemma 9 *Let $T \subseteq M$ be a zerofree subset with $\#T = 4$. Then $\Delta(T) \geq 9$.*

Proof. Let $T = \{t_1, t_2, t_3, t_4\}$. If each t_i is a subset sum of the remaining t_j we are in the second case of Lemma 8 and conclude that $\Delta(T) \geq 2r+1 = 9$. Therefore (without restriction) we may assume that t_4 is not a subset sum of $T' = \{t_1, t_2, t_3\}$ and $\Delta(T) \geq \Delta(T') + 2$. If $\Delta(T') \geq 7$ we are done. So we may assume that $\Delta(T') = 6$. We have five different subset sums

$$0,\ t_1,\ t_2,\ t_3,\ t_1+t_2+t_3,$$

and the sixth one must be a two-element sum, say

$$t_1 + t_2.$$

Then necessarily $t_1 + t_3 = t_2$ (no other one of the six subset sums fits here), and likewise $t_2 + t_3 = t_1$. Addition of these two relations yields $2\,t_3 = 0$ or $t_3 = -t_3$.

Now considering t_4 we get two additional subset sums

$$t_4 \quad \text{and} \quad \Sigma(T) = t_1 + t_2 + t_3 + t_4$$

and need only one more. The sum

$$t_1 + t_2 + t_4$$

is certainly different from the seven sums 0, t_1, t_2, t_4, t_1+t_2, $t_1+t_2+t_3$, and $\Sigma(T)$. Could it be $= t_3$? Since $t_3 = -t_3$ this yields $t_1 + t_2 + t_4 = -t_3$, thus $t_1 + t_2 + t_3 + t_4 = 0$, contradiction.

Therefore $\Delta(T) \geq 9$. ◇

6 Olson's Theorem

A special case of Olson's results in [10] leads to the strongest known improvement of the Eggleton-Erdős bound $2r + 1$.

Theorem 4 *Let M be an abelian group and $T \subseteq M$ be a finite subset, $r = \#T$. Then at least one of the following statements holds:*

(i) *For each subset* $S \subseteq T$ *there is another subset* $U \subseteq T$ *with* $U \neq S$ *and* $\Sigma(U) = \Sigma(S)$.

(ii) $\Delta(T) \geq 1 + r^2/9$.

In fact Olson proved this (with an appropriate formulation) even for nonabelian M [10, Theorem 3.2]. We omit the proof.

If $T \subseteq M$ is zerofree, it violates statement (i) for the subset sum $0 = \Sigma(\emptyset)$. Hence T must satisfy statement (ii):

Corollary 1 *Let* M *be an abelian group and* $T \subseteq M$ *be a finite zerofree subset,* $r = \#T$. *Then* $\Delta(T) \geq 1 + r^2/9$ *(as long as this number is* $\leq \#M$*).*

Corollary 2 *Let* M *be an abelian group and* $r \geq 1$. *Then*

$$\Delta_M(r) \geq 1 + \left\lceil \frac{r^2}{9} \right\rceil$$

(as long as this number is $\leq \#M$*).*

This bound is larger than $2r + 1$ if and only if $r \geq 19$.

Note 1 Call $T \subseteq M$ **antisymmetric** if $T \cap (-T) = \emptyset$.

- Let $M = \mathbb{Z}/p\mathbb{Z}$, p a prime. Let T be an antisymmetric subset of M of size $r = \#T$. Then Balandraud [1] improved a bound by Olson [9] as follows:

$$\Delta(T) \geq \min \left\{ p,\ 1 + \frac{r\,(r+1)}{2} \right\}.$$

- Let M be a finite abelian group of order $m = \#M$. Let $T \subseteq M$ antisymmetric of size $r = \#T \geq 2$. Then at least one the following statements is true [2]:
  (i) $\Delta(T) \geq \frac{r\,(r-1)}{2} + 3$.
  (ii) There is nonempty subset $U \subseteq T$ with $\Delta(U) > \#\langle U \rangle/2$.

  For odd m the first item may be replaced by $\Delta(T) \geq \frac{r\,(r+1)}{2}$.

Remark If T is not antisymmetric, then it contains an element $t \in T \cap (-T)$, $t = -s$ with $s \in T$, thus $s + t = 0$. We distinguish the cases:

- $t = 0$. Then T has the zero-sum subset $\{0\}$ of size 1.
- t has order 2. Then $s = t$.
- $t \neq 2$. Then T has the zero-sum subset $\{s, t\}$ of size 2.

We conclude that if T is zerofree and doesn't contain an element of order 2, then T is antisymmetric.

Note 2 Let $M = \mathbb{Z}/p\mathbb{Z}$, p a prime ≥ 3. Then M doesn't contain an element of order 2. Therefore for a zerofree subset $T \subseteq M$ of size $r = \#T$ Note 1 yields the bound:

$$\Delta(T) \geq 1 + \frac{r\,(r+1)}{2}$$

as long as this number is $\leq p$. As in Section 1 this bound is sharp. Using Lemma 2 we conclude that

$$\mathsf{zf}(\mathbb{Z}/p\mathbb{Z}) = \max\left\{ r \;\middle|\; \frac{r\,(r+1)}{2} < p \right\} = \left\lceil \sqrt{2p + \frac{1}{4}} - \frac{3}{2} \right\rceil .$$

Note 3 A strong variant of the Erdős-Heilbronn conjecture claims that $\mathsf{zf}(\mathbb{Z}/m\mathbb{Z}) < \lceil \sqrt{2m}\, \rceil$. In other words, if $\mathbb{Z}/m\mathbb{Z}$ has a zerofree subset of size r, then $m > r^2/2$. For $m = p$ prime Note 2 implies the slightly stronger inequality $\mathsf{zf}(\mathbb{Z}/p\mathbb{Z}) < \sqrt{2p-1}$.

7 Subset Sums $\bmod m$

In this section we consider the group $M = \mathbb{Z}/m\mathbb{Z}$ and write Δ_m instead of Δ_M. We freely abuse the notation by identifying the integer $a \in \{0, \ldots, m-1\}$ with its residue class $\bmod m$.

Example 3. For $m \leq 5$ zerofree subsets $T \subseteq M$ have $r \leq 2$ elements, hence all possible values $\Delta(T)$ are known (and depend only on $\#T$, see Table 2). We have $\Delta_m(1) = 2$ and $\Delta_m(2) = 4$.

Example 4. For $m = 6$ and $r = 3$ there are two zerofree three-element subsets T of $\{1, 2, 3, 4, 5\}$:

$$T(6, 1) = \{1, 3, 4\} \quad \text{and} \quad T(6, 2) = \{2, 3, 5\}.$$

For each of them $\Delta(T) = 6 = m$. Thus $\Delta_6(3) = 6$.

Example 5, $m = 7$ and $r = 3$. There are six zerofree three-element subsets T of $\{1, 2, 3, 4, 5, 6\}$:

$$\{1, 2, 3\},\ \{1, 3, 5\},\ \{1, 4, 5\},\ \{2, 3, 6\},\ \{2, 4, 6\},\ \{4, 5, 6\}.$$

For each of them $\Delta(T) = 7 = m$. Thus $\Delta_7(3) = 7$.

Example 6, $m = 8$. The cases $r = 0, 1, 2$ are known. In the case $r = \#T = 3$ the picture is inhomogeneous:

- The subset $T = \{1, 4, 5\} \subseteq \{1, 2, 3, 4, 5, 6, 7\}$ is zerofree and has $\Delta(T) = 6$.

- The subset $T = \{1, 2, 3\}$ is zerofree and has $\Delta(T) = 7$.
- The subset $T = \{1, 4, 6\}$ is zerofree and has $\Delta(T) = 8$.

Thus each of the possible values $\Delta(T) = 6, 7, 8$ occurs, and $\Delta_8(3) = 6$.

Note Balandraud's bound, see Note 2 in Section 6 yields the exact values for prime modules $m = p$:

$$\Delta_p(r) = 1 + \frac{r\,(r+1)}{2}$$

(as long as this value is $\leq p$, that is $\mathsf{zf}(\mathbb{Z}/p\mathbb{Z}) \leq r$). This formula yields for example:

- $\Delta_p(1) = 1 + 1 \cdot 2/2 = 2$ for all $p \geq 3$.
- $\Delta_p(2) = 1 + 2 \cdot 3/2 = 4$ for all $p \geq 5$.
- $\Delta_p(3) = 1 + 3 \cdot 4/2 = 7$ for all $p \geq 7$.
- $\Delta_p(4) = 1 + 4 \cdot 5/2 = 11$ for all $p \geq 11$.
- $\Delta_p(5) = 1 + 5 \cdot 6/2 = 16$ for all $p \geq 17$.
- $\Delta_p(6) = 1 + 6 \cdot 7/2 = 22$ for all $p \geq 23$.

Table 2 summarizes the known values for small modules m and extends this knowledge with the help of a Python (or SageMath) program, listed in Appendix B. The table exhibits some eye-catching patterns that suggest several hypotheses:

1. For $r \geq 3$ the value $\Delta_m(T) = 2r$ occurs only if m is even and $r = 3$. This follows from Theorem 3 (ii) and Corollary 1 of Proposition 1.

2. For $r = 3$ the possible diversities are

$$\Delta(T) = \begin{cases} 6,\ 7,\ 8 & \text{if } m \text{ is even,} \\ 7,\ 8 & \text{if } m \text{ is odd.} \end{cases}$$

   Therefore

$$\Delta_m(3) = \begin{cases} 6 & \text{if } m \text{ is even,} \\ 7 & \text{if } m \text{ is odd.} \end{cases}$$

   This was proved in Section 3.

3. For $r = 4$ zerofree sets exist only if $m \geq 9$, and then always $\Delta(T) \geq 9 = 2r + 1$, and $\Delta_m(4) \geq 9$. The theoretical maximum diversity $\min(m, 16)$ is reached. For $m \geq 10$ we see even $\Delta(T) \geq 10 = 2r+2$, hence $\Delta_m(4) \geq 10$, even ≥ 11 if m is prime, confirming the note above.

$r = \#T =$	0	1	2	3	4	5	6	7	8
$m = 1$	1	–	–	–	–	–	–	–	–
2	1	2	–	–	–	–	–	–	–
3	1	2	–	–	–	–	–	–	–
4	1	2	4	–	–	–	–	–	–
5	1	2	4	–	–	–	–	–	–
6	1	2	4	6	–	–	–	–	–
7	1	2	4	7	–	–	–	–	–
8	1	2	4	6..8	–	–	–	–	–
9	1	2	4	7..8	9	–	–	–	–
10	1	2	4	6..8	10	–	–	–	–
11	1	2	4	7..8	11	–	–	–	–
12	1	2	4	6..8	10..12	–	–	–	–
13	1	2	4	7..8	11..13	–	–	–	–
14	1	2	4	6..8	10..14	14	–	–	–
15	1	2	4	7..8	10..15	15	–	–	–
16	1	2	4	6..8	10..16	14..16	–	–	–
17	1	2	4	7..8	11..16	16..17	–	–	–
18	1	2	4	6..8	9..16	14, 16..18	–	–	–
19	1	2	4	7..8	11..16	16..19	–	–	–
20	1	2	4	6..8	10..16	14, 16..20	20	–	–
21	1	2	4	7..8	10..16	16..21	21	–	–
22	1	2	4	6..8	10..16	14, 16..22	20, 22	–	–
23	1	2	4	7..8	11..16	16..23	22, 23	–	–
24	1	2	4	6..8	10..16	14, 16..24	20 22..24	–	–
25	1	2	4	7..8	11..16	16..25	22..25	25	–
26	1	2	4	6..8	10..16	14, 16..26	20 22..26	26	–

Table 2: Possible values of $\Delta(T)$, T zerofree, for small m.

4. For $r = 5$ zerofree sets exist only if $m \geq 14$. The possible diversities are
$$\Delta(T) \geq \begin{cases} 14 & \text{if } m \text{ is even,} \\ 15 & \text{if } m = 15, \\ 16 & \text{if } m \text{ is odd} \geq 17. \end{cases}$$
In particular $\Delta_m(5) \geq 14 = 2r+4$, even ≥ 16 if m is prime. The value 15 doesn't occur as $\Delta(T)$ for even $m \geq 18$ nor for odd $m \geq 17$.

5. For $r = 6$ zerofree sets exist only if $m \geq 20$. The possible diversities are
$$\Delta(T) \geq \begin{cases} 20 & \text{if } m \text{ is even,} \\ 21 & \text{if } m = 21, \\ 22 & \text{if } m \text{ is odd} \geq 23. \end{cases}$$
In particular $\Delta_m(6) \geq 20 = 2r+8$. There is no zerofree set of diversity 21 for even m, or for odd $m \geq 21$.

6. All zerofree subsets $T \subseteq \mathbb{Z}/m\mathbb{Z}$ have size
$$\leq \ \mathsf{zf}(\mathbb{Z}/m\mathbb{Z}) = \begin{cases} 3 & \text{for } 6 \leq m \leq 8, \\ 4 & \text{for } 9 \leq m \leq 13, \\ 5 & \text{for } 14 \leq m \leq 19, \\ 6 & \text{for } 20 \leq m \leq 24. \end{cases}$$

Thus Table 2 suggests that the lower bound of Lemma 2 is at most one less then zf for composite modules m, and zf seems to be monotone in m.

We know that
$$\Delta_m(0) = 1, \quad \Delta_m(1) = 2, \quad \Delta_m(2) = 4, \quad \Delta_m(3) = 6, \quad \Delta_m(4) = 9,$$
and, from [3], that
$$\Delta_m(5) = 14, \quad \Delta_m(6) = 20, \quad \Delta_m(7) = 25.$$
Further calculations suggest that $\Delta_m(8) \geq 34$ (for m prime the value is 37).

8 Some Further Remarks

For an abelian group M and a subset $T \subseteq M$ let
$$\mathcal{S}^*(T) = \{\Sigma(U) \mid U \subseteq T,\, U \neq \emptyset\}$$
be the set of all nontrivial subset sums of T. Note that $\mathcal{S}(M) = \mathcal{S}^*(M) = M$ and $\mathcal{S}^*(M - \{0\}) \supseteq M - \{0\}$.

Lemma 10 *$\mathcal{S}^*(M - \{0\}) = M$ if and only if $\#M \geq 3$.*

Proof. If $\#M = 1$, then $M - \{0\} = \emptyset$, thus $\mathcal{S}^*(M - \{0\}) = \emptyset$.

If $\#M = 2$, say $M = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$, then $\mathcal{S}^*(M - \{0\}) = \{1\}$.

Now let $\#M \geq 3$. Then there are $t_1, t_2 \in M - \{0\}$ with $t_1 \neq t_2$. If $t_1 + t_2 = 0$, then $0 \in \mathcal{S}^*(M - \{0\})$, and we are done. Otherwise $t_1 + t_2 = t_3 \neq 0$. If $t_3 = -t_1$, then $\{t_1, t_3\}$ sums to zero; likewise if $t_3 = -t_2$. In the remaining case $\{t_1, t_2, -t_3\}$ is a three-element subset with sum 0. $\diamond$

Definition The integer

$$\mathsf{cr}(M) = \min\{r \mid \mathcal{S}^*(T) = M \text{ for all } T \subseteq M - \{0\} \text{ with } \#T \geq r\},$$

is called the **covering** (or critical) **constant** of M.

Example 1 If $\#M \leq 2$ by Lemma 10 M is never covered, thus $\mathsf{cr}(M) = \infty$.

Example 2 Let $M = \mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\}$ (where by abuse of notation we let integers represent their own residue classes).

- For $T = \{1\}$ we have $\Sigma(T) = 1$, $\mathcal{S}(T) = \{0, 1\}$, $\mathcal{S}^*(T) = \{1\}$. As a consequence $\mathsf{cr}(M) > 1$.
- For $T = \{1, 2\}$ we have $\Sigma(T) = 0$, $\mathcal{S}(T) = \mathcal{S}^*(T) = \{0, 1, 2\}$. Since T is the only two-element subset of $M - \{0\}$ we have $\mathsf{cr}(M) = 2$.

Example 3 Let $M = \mathbb{Z}/4\mathbb{Z} = \{0, 1, 2, 3\}$.

- Let $a \neq 0$. For $T = \{a\}$ we have $\Sigma(T) = a$, $\mathcal{S}(T) = \{0, a\}$, $\mathcal{S}^*(T) = \{a\}$. This implies $\mathsf{cr}(M) > 1$.
- Let $a, b \in M - \{0\}$, $a \neq \pm b$. For $T = \{a, b\}$ we have $\Sigma(T) = a + b \neq 0$, $\mathcal{S}(T) = \{0, a, b, a + b\} = M$, $\mathcal{S}^*(T) = \{a, b, a + b\} = M - \{0\}$.
- For $T = \{1, 3\}$ we have $\Sigma(T) = 0$, $\mathcal{S}(T) = \{0, 1, 3\}$, $\mathcal{S}^*(T) = \{0, 1, 3\}$. Therefore $\mathcal{S}^*(T) \neq M$ for all two-element subsets $T \subseteq M - \{0\}$, thus $\mathsf{cr}(M) > 2$.
- For $T = \{1, 2, 3\} = M - \{0\}$ we have $\Sigma(T) = 2$, $\mathcal{S}(T) = M$, $\mathcal{S}^*(T) = M$. Therefore $\mathsf{cr}(M) = 3$.

Note By [5] we have $\mathsf{cr}(\mathbb{Z}/p\mathbb{Z}) = \lfloor 2\sqrt{p-2} \rfloor$. The covering constant $\mathsf{cr}(M)$ is known for all finite abelian groups M, see [7]: Let p be the smallest prime divisor of $m = \#M$ and $m \neq p$. Then

$$\mathsf{cr}(M) = \frac{m}{p} + p - \delta$$

where $\delta = 2$ (in the general case) or $\delta = 1$ (in a small, explicitly known set of exceptional cases).

Definition We call $U \subseteq M$ a **minimal** zero-sum subset if $U \neq \emptyset$, $\Sigma(U) = 0$, and U is minimal under these conditions. The **strong Davenport constant** $\mathsf{SD}(T)$ of $T \subseteq M$ is the maximum size of a minimal zero-sum subset of T, see [4].

Definition **Olson's constant** $\mathsf{Ol}(M)$ is the smallest r such that each subset $T \subseteq M$ of size r contains a nontrivial zero-sum subset:

$$\mathsf{Ol}(M) = \min\{r \mid 0 \in \mathcal{S}^*(T) \text{ for all } T \subseteq M \text{ with } \#T \geq r\}.$$

Lemma 11 *$\mathsf{SD}(M) \leq \mathsf{Ol}(M) \leq \mathsf{cr}(M)$.*

Proof. For the first inequality we have to show that $\#T \leq \mathsf{Ol}(M)$ for an arbitrary minimal zero-sum set T. Assuming $\#T > \mathsf{Ol}(M)$ we take $T' = T - \{t\}$ for an arbitrary $t \in T$ and conclude that $\#T' \geq \mathsf{Ol}(M)$, hence $0 \in \mathcal{S}^*(T')$, $0 = \Sigma(U)$ for some nonempty subset $U \subseteq T' \subset T$, contradicting the minimality of T.

The second inequality is trivial if $\mathsf{cr}(M) = \infty$. If $\mathsf{cr}(M) < \infty$ we have to show that $0 \in \mathcal{S}^*(T)$ if $\#T \geq \mathsf{cr}(M)$. If $0 \in T$ this is trivial. Otherwise $T \subseteq M - \{0\}$, hence $M = \mathcal{S}^*(T)$ by the definition of cr, a forteriori $0 \in \mathcal{S}^*(T)$. $\diamond$

Lemma 12 *$\mathsf{zf}(M) = \mathsf{Ol}(M) - 1$.*

Proof. We may assume that both quantities are finite.

"$\leq$": Take a zerofree set T of maximal size $\#T = \mathsf{zf}(M)$. Then obviously $\#T < \mathsf{Ol}(M)$.

"$\geq$": There is a subset $T \subseteq M$ of size $\#T = \mathsf{Ol}(M) - 1$ such that $0 \notin \mathcal{S}^*(T)$. Hence T is zerofree, $\mathsf{Ol}(M) - 1 = \#T \leq \mathsf{zf}(M)$. $\diamond$

Example 1 If $\#M \leq 2$ the only zero-sum subset is $\{0\}$. Thus $\mathsf{SD}(M) = 1$, and $\mathsf{Ol}(M) = \#M = 1$ or 2. The maximal zerofree subsets are $\emptyset$ if $M = \{0\}$, and $\{1\}$ if $M = \mathbb{Z}/2\mathbb{Z}$, hence $\mathsf{zf}(M) = 0$ or 1.

Example 2 Let $M = \mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\}$. The $8 = 2^3$ different subsets are

$$\emptyset,\ \{0\},\ \{1\},\ \{2\},\ \{0,1\},\ \{0,2\},\ \{1,2\},\ \{0,1,2\}.$$

- Zero-sum subsets: $\emptyset$, $\{0\}$, $\{1,2\}$, $\{0,1,2\}$. Thus $\mathsf{Ol}(M) = 2$.
- Minimal zero-sum subsets: $\{0\}$, $\{1,2\}$. Thus $\mathsf{SD}(M) = 2$.
- Zerofree subsets: $\{1\}$, $\{2\}$. Thus $\mathsf{zf}(M) = 1$.

Example 3 Let $M = \mathbb{Z}/4\mathbb{Z} = \{0, 1, 2, 3\}$. The $16 = 2^4$ different subsets are

$$\emptyset,\ \{0\},\ \{1\},\ \{2\},\ \{3\},\ \{0,1\},\ \{0,2\},\ \{0,3\},\ \{1,2\},\ \{1,3\},\ \{2,3\},$$

$$\{0,1,2\},\ \{0,1,3\},\ \{0,2,3\},\ \{1,2,3\},\ \{0,1,2,3\}.$$

- Zero-sum subsets: $\emptyset$, $\{0\}$, $\{1,3\}$, $\{0,1,3\}$. Thus $\mathsf{OI}(M) = 3$.
- Minimal zero-sum subsets: $\{0\}$, $\{1,3\}$. Thus $\mathsf{SD}(M) = 2$.
- Zerofree subsets: $\{1\}$, $\{2\}$, $\{3\}$, $\{1,2\}$, $\{1,3\}$. Thus $\mathsf{zf}(M) = 2$.

For the comparision of zf and SD it's convenient to consider also multisets.

Proposition 2 *Let S be a zerofree multiset in a $\mathbb{Z}$-module M. Then the number $\mathsf{w}(S)$ of different elements of S is at most $\mathsf{SD}(M)$.*

Proof. By definition $t := -\Sigma(S) \in M - \{0\}$, hence $T := S \cup \{t\}$ is a zerosum multiset, $\Sigma(T) = \Sigma(S) + t = 0$. There is a minimal zerosum multiset $U \subseteq T$. Since S is zerofree U is not contained in S, hence the multiplicity of t in U is $1+$ the multiplicity of t in S, and $U' := U - \{t\}$ (multiplicity of t decreased by 1) is a submultiset of S. Moreover

$$\Sigma(U') = \Sigma(U) - t = -t = \Sigma(S).$$

Therefore $S - U'$ is a zerosum multiset contained in S, hence $= \emptyset$, thus $U' = S$ and $U = U' \cup \{t\} = S \cup \{t\} = T$. Since U is minimal $\mathsf{w}(S) \leq \mathsf{w}(T) = \mathsf{w}(U) \leq \mathsf{SD}(M)$. $\diamond$

Corollary 1 *If $S \subseteq M$ is a zerofree subset, then $\#S \leq \mathsf{SD}(M)$. In particular $\mathsf{zf}(M) \leq \mathsf{SD}(M)$.*

Proof. Since S is a set $\#S = \mathsf{w}(S)$. $\diamond$

Corollary 2 *Assume $\mathsf{SD}(M) < \infty$. Then $\mathsf{zf}(M) = \mathsf{SD}(M)$ or $\mathsf{SD}(M) - 1$, and $\mathsf{SD}(M) = \mathsf{OI}(M)$ or $\mathsf{OI}(M) - 1$.*

Proof. $\mathsf{zf}(M) \leq \mathsf{SD}(M)$ by Corollary 1. To get a zerofree set of size $\mathsf{SD}(M) - 1$ take a minimal zero-sum subset of size $\mathsf{SD}(M)$ and remove an arbitrary element. The second statement follows from Lemma 12. $\diamond$

In summary we have

Corollary 3 *Let M be an abelian group. Then*

$$\mathsf{zf}(M) \overset{(1)}{\leq} \mathsf{SD}(M) \overset{(1)}{\leq} \mathsf{OI}(M) = \mathsf{zf}(M) + 1 \leq \mathsf{cr}(M)$$

where exactly one of the inequalities (1) *or* (2) *is an equality.*

Note that all these numbers are defined for arbitrary M but make sense only for M finite.

Note From [1] for $M = \mathbb{Z}/p\mathbb{Z}$ we have

$$\mathsf{Ol}(\mathbb{Z}/p\mathbb{Z}) = \min \left\{ k \;\middle|\; \frac{k\,(k+1)}{2} \geq p \right\}.$$

By Note 2 in Section 6 $\mathsf{zf}(\mathbb{Z}/p\mathbb{Z}) < \sqrt{2p-1}$. Hence by Corollary 3

$$\mathsf{SD}(\mathbb{Z}/p\mathbb{Z}) \leq \mathsf{Ol}(\mathbb{Z}/p\mathbb{Z}) \leq \left\lceil \sqrt{2p-1} \right\rceil.$$

Olson's bound from [9] was $\mathsf{Ol}(\mathbb{Z}/p\mathbb{Z}) \leq \mathsf{cr}(\mathbb{Z}/p\mathbb{Z}) \leq \lceil \sqrt{4p-3} \rceil$. From [5] we know the slighly smaller bound

$$\mathsf{cr}(\mathbb{Z}/p\mathbb{Z}) = \left\lfloor 2\sqrt{p-2} \right\rfloor.$$

Figures 1 and 2 illustrate the growth of zf and of the number of minimal zero-sum sets as a function of the module m, generated with the program from Appendix C.

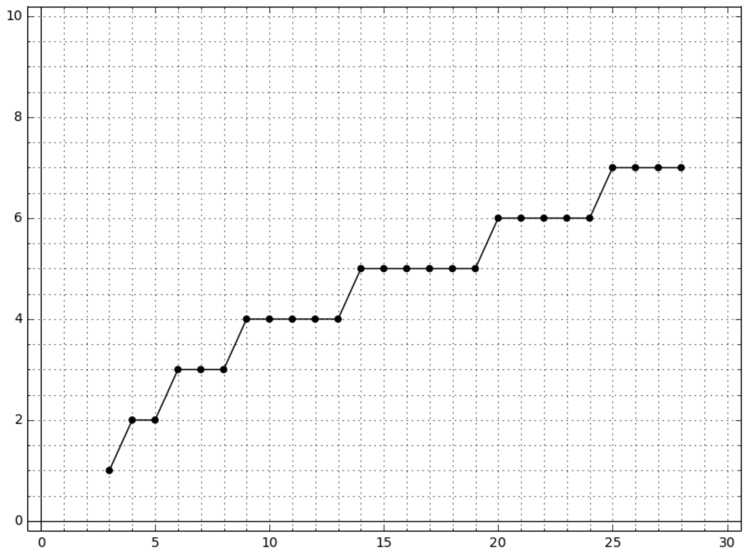

Figure 1: The zerofree bound of $\mathbb{Z}/m\mathbb{Z}$

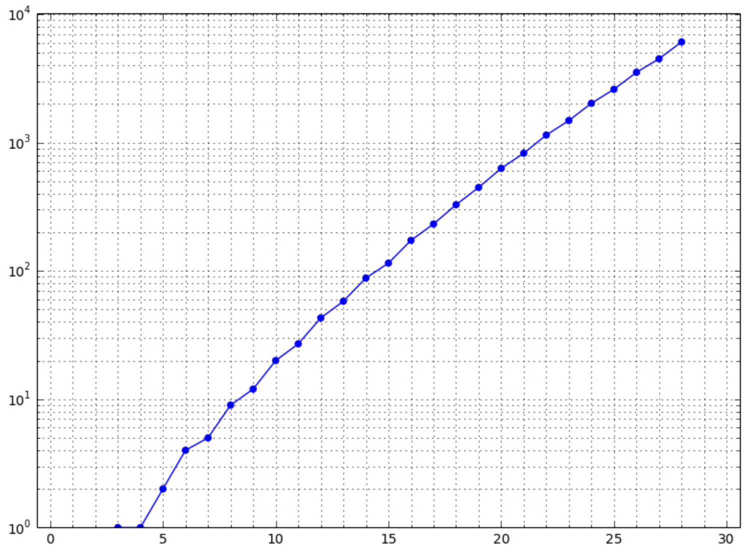

Figure 2: The number of minimal zero-sum sets mod m

A Auxiliary Routines in Python

Calculate the list of coefficients for the representation on s in base b

```
def baserep(s,b):
  coefflist = []
  while s != 0:
    rem = s % b
    quot = s//b
    coefflist.insert(0,rem)
    s = quot
  return coefflist
```

Calculate all subsets of a set

```
def subsets(T):
  ll = len(T)
  NN = 2**ll
  bitlist = []
  subsetlist = []
  L = list(T)
  for i in range(NN):           # build selection scheme for elements
    bitvector = baserep(i,2)
    while len(bitvector) < ll:
      bitvector.insert(0,0)
    bitlist.append(bitvector)
  for bitvector in bitlist:     # construct corresponding subset
    newlst = []
    for j in range(ll):
      if bitvector[j] == 1:
        newlst.append(L[j])
    newset = set(newlst)
    subsetlist.append(newset)
  return(subsetlist)
```

B Zerofree Subsets and Diversities in Python

```
m = int(sys.argv[1])
base = list(range(1,m))
print("Module:", m)

stoplist = []           # Supersets are not zerofree.
list1 = []              # List of zerofree subsets
for i in range(1,m):
  list1.append({i})

mm = 1 + m//2
for r in range(2,mm):
  print("r =", r, "| Zerofree r-element sets:")
  list2 = []            # Next list of zerofree subsets
  divlist = []          # List of diversities
  for S in list1:
    tm = max(S)
    for i in range(tm+1,m): # Add on element to the set S.
      T = S.copy()
      T.add(i)
      stopcond = False
      for stopset in stoplist:
        if stopset <= T:
          stopcond = True
      if not(stopcond):
        ss = sum(T) % m
        if ss == 0:
          stoplist.append(T)
        else:
          list2.append(T)
          sublist = subsets(T)    # Now calculate all subset sums of T
          sumlist = []
          for U in sublist:
            sumlist.append(sum(U) % m)
          sumset = set(sumlist)
          Delta = len(sumset)
          divlist.append(Delta)
  list1 = list2.copy()   # Save list for use in next round.
  if len(divset) > 0:
    print("r =", r, "| Diversities:", divset)
  else:
    print("r =", r, "| No zerofree sets")
```

C Strong Davenport Constant and Number of Minimal Zero-sum Sets

```
mm = int(sys.argv[1])
zslist = []  # list of minimal zerosum subsets, to be built successively
zflist = []  # list of zerofree subsets of actual size,
             #   to be replaced in each step

for t in range(1,mm):
  zflist.append({t})

zf = 1                        # zerofree bound
SD = 1                        # strong Davenport constant
s = 1                         # actual size
while len(zflist) > 0:        # stop condition not yet reached
  s += 1                      # next size
  oldlist = zflist.copy()     # zerofree sets of previous size
  zflist = []                 # zerofree sets of actual size
  for oldset in oldlist:      # expand each zerofree set
    for t in range(1,mm):     #   by one element t
      discard = False
      newset = oldset.copy()
      newset.add(t)
      if len(newset) < s or newset in zflist:
        discard = True        # discard if t already in oldset
                              #   or newset not really new
      else:
        for zsset in zslist: # or if newset contains a zerosum set
          if zsset <= newset:
            discard = True
      if not(discard):
        if sum(newset) % mm == 0: # test zerosum property
          zslist.append(newset)   # new minimal zerosum subset detected
          SD = s     # update value for strong Davenport constant
        else:
          zflist.append(newset)   # new zerofree subset detected
          zf = s
print("m:", mm, "| zf = ", zf, "| SD = ", SD, "| zs = ", len(zslist))
```

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